Construction of positive integers with even period of minimal type

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Abstract: We give a construction of positive integers with even period of minimal type.

Key words: Real quadratic fields of minimal type; continued fractions.

Introduction. In a previous paper [3], for any positive integer ℓ, we introduced the notion of *real quadratic fields with period ℓ of minimal type* (see Definition below) by using the simple continued fraction expansions with period ℓ of certain quadratic irrationals, and proved that there exist exactly 51 real quadratic fields of class number 1 that are not of minimal type, with one more possible exception ([3, Proposition 4.4]). On the other hand, Sasaki [7] and Lachaud [6] showed that for any positive integers ℓ and h, there exist at most finitely many real quadratic fields with period ℓ of class number h. Hence we have to examine a construction of real quadratic fields with non-fixed period ℓ of minimal type in order to find many real quadratic fields of class number 1. In [4, Theorem 1.1] we proved, in the case where $\ell \geq 4$ is an even integer with $8 \nmid \ell$, that there exist infinitely many real quadratic fields with period ℓ of minimal type. In this paper, for any even integer $\ell \geq 4$, we shall examine a construction of *positive integers with period ℓ of minimal type* (see Definition below; Theorem 2) in a different way, as the first step of getting real quadratic fields with period ℓ of minimal type. To see a usefulness of Theorem 2, we show the infiniteness of real quadratic fields with period 8 of minimal type (Example 2).

1. Positive integers of minimal type. Let $\ell \geq 2$ be a positive integer and let $a_1, a_2, \dots, a_{\ell-1}$ be a symmetric string of $\ell - 1$ positive integers. We define nonnegative integers q_n, r_n by using a_n:

$$(1.1)\quad \begin{cases} q_0 = 0,\ q_1 = 1,\ q_n = a_{n-1}q_{n-1} + q_{n-2}\ (n \geq 2), \\ r_0 = 1,\ r_1 = 0,\ r_n = a_{n-1}r_{n-1} + r_{n-2}\ (n \geq 2). \end{cases}$$

Then we easily see that $q_n \geq r_n\ (n \geq 1)$. For brevity, we put

$$A := q_\ell, \quad B := q_{\ell-1}, \quad C := r_{\ell-1},$$

and define polynomials $g(x), h(x), f(x)$ by

$$g(x) = Ax - (-1)^\ell BC, \quad h(x) = Bx - (-1)^\ell C^2,$$
$$f(x) = g(x)^2 + 4h(x).$$

Furthermore, let s_0 be the least integer x for which $x > (-1)^\ell BC/A$. We consider three cases separately:

(I) $A \equiv 1 \pmod 2$, (II) $(A, C) \equiv (0, 0) \pmod 2$,
(III) $(A, C) \equiv (0, 1) \pmod 2$.

The following theorem was shown in [3, Theorem 3.1] which is an improvement of results of Friesen [1, Theorem] and of Halter-Koch [2, Theorem 1A, Corollary 1A]. For a real number x, we denote by $[x]$ the largest integer $\leq x$.

Theorem 1. *Let $\ell \geq 2$ be a fixed positive integer and $a_1, \dots, a_{\ell-1}$ any symmetric string of $\ell - 1$ positive integers.*

When Case (I) or Case (II) occurs, we let s be any integer with $s \geq s_0$, and put $d := f(s)/4$ and $a_0 := g(s)/2$. Here, we choose an even integer s in Case (I), and assume that

$$(1.2)\qquad g(s) > a_1, \dots, a_{\ell-1}.$$

Then, d and a_0 are positive integers, d is non-square, $a_0 = [\sqrt{d}]$ and the simple continued fraction expansion of $\sqrt{d}$ is

$$(1.3)\qquad \sqrt{d} = [a_0, \overline{a_1, \dots, a_{\ell-1}, 2a_0}]$$

with minimal period ℓ. Also, in Case (III), there is no positive integer d such that (1.3) is the simple continued fraction expansion of $\sqrt{d}$.

When Case (I) or Case (III) occurs, we let s be any integer with $s \geq s_0$, and put $d := f(s)$ and

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$a_0 := (g(s)+1)/2$. *Here, we choose an odd integer* s *in Case* (I), *and assume that* (1.2) *holds. Then,* d *and* a_0 *are positive integers,* d *is non-square,* $d \equiv 1 \pmod 4$, $a_0 = [(1+\sqrt{d})/2]$ *and the simple continued fraction expansion of* $(1+\sqrt{d})/2$ *is*

$$(1.4) \qquad \frac{1+\sqrt{d}}{2} = [a_0, \overline{a_1, \dots, a_{\ell-1}, 2a_0 - 1}]$$

with minimal period ℓ. *Also, in Case* (II), *there is no positive integer* d *such that* $d \equiv 1 \pmod 4$ *and* (1.4) *is the simple continued fraction expansion of* $(1+\sqrt{d})/2$.

Conversely, we let d *be any non-square positive integer. By using a quadratic polynomial* $f(x)$ *and an integer* s_0 *obtained as above from the symmetric part of the simple continued fraction expansion of* $\sqrt{d}$, d *becomes uniquely of the form* $d = f(s)/4$ *with some integer* $s \geq s_0$, *and* (1.2) *holds. If* $d \equiv 1 \pmod 4$ *in addition then the same thing is true for* $(1+\sqrt{d})/2$.

Definition ([3, Definition 3.1])**.** Let d be a non-square positive integer. By Theorem 1, d is uniquely of the form $d = f(s)/4$ with some integer $s \geq s_0$, where $f(x)$ and s_0 are obtained as above from the symmetric part $a_1, a_2, \dots, a_{\ell-1}$ of the simple continued fraction expansion of $\sqrt{d}$ and ℓ is the minimal period. If $s = s_0$, that is, $d = f(s_0)/4$ holds, then we say that d is a *positive integer with period* ℓ *of minimal type for* (*the simple continued fraction expansion of*) $\sqrt{d}$. When $d \equiv 1 \pmod 4$ in addition, d is uniquely of the form $d = f(s)$ with some integer $s \geq s_0$, where $f(x)$ and s_0 are obtained as above from the symmetric part $a_1, a_2, \dots, a_{\ell-1}$ of the simple continued fraction expansion of $(1+\sqrt{d})/2$ and ℓ is the minimal period. If $s = s_0$, that is, $d = f(s_0)$ holds, then we say that d is a *positive integer with period* ℓ *of minimal type for* (*the simple continued fraction expansion of*) $(1+\sqrt{d})/2$.

Furthermore, for a square-free positive integer $d > 1$, we say that $\mathbf{Q}(\sqrt{d})$ is a *real quadratic field with period* ℓ *of minimal type*, if d is a positive integer with period ℓ of minimal type for $\sqrt{d}$ when $d \equiv 2, 3 \pmod 4$, and if d is a positive integer with period ℓ of minimal type for $(1+\sqrt{d})/2$ when $d \equiv 1 \pmod 4$.

In the next section, following [3], [4] and [5], we calculate s_0, $g(s_0)$, $h(s_0)$ in order to get positive integers with even period of minimal type.

2. Calculation of s_0, $g(s_0)$ and $h(s_0)$. Fix a string of L (≥ 2) positive integers $a_1, \dots, a_L$ and put $\ell = 2L$. Define

$$a_{L+1} := a_{L-1},\ a_{L+2} := a_{L-2}, \dots, a_{2L-1} := a_1;$$

we get a symmetric string $a_1, \dots, a_{\ell-1}$ of $\ell - 1$ positive integers. From this string, we define non-negative integers q_n, r_n by (1.1). Then it is well-known that

$$(2.1) \qquad q_n r_{n-1} - q_{n-1} r_n = (-1)^{n-1} \ (1 \leq n \leq \ell).$$

Moreover the following hold.

Lemma. *Under the above setting, we have*

$$A = q_\ell = (q_{L+1} + q_{L-1}) q_L,$$
$$B = q_{\ell-1} = (q_{L+1} + q_{L-1}) r_L - (-1)^L,$$
$$C = r_{\ell-1} = (r_{L+1} + r_{L-1}) r_L,$$
$$B^2 - AC = (-1)^\ell = 1.$$

Proof. See [4, Lemma 2.2 (i)] and [4, (2.8)]. □

Now we define integers u_1, u_2, w, v_1, v_2, z, δ by

$$(2.2) \qquad (r_L^2 - (-1)^L)(r_{L+1} + r_{L-1}) = q_L v_1 + u_1 \ (0 \leq u_1 < q_L),$$

$$(2.3) \qquad (-1)^L (r_L - q_{L-1}) r_L = q_L z + w \ (0 \leq w < q_L),$$

$$(2.4) \qquad (-1)^L (q_L - r_{L+1}) + z = q_L v_2 + u_2 \ (0 \leq u_2 < q_L),$$

$$\delta = \begin{cases} 0 & \text{if } u_1 \leq u_2, \\ 1 & \text{if } u_1 > u_2, \end{cases}$$

and calculate s_0, $g(s_0)$, $h(s_0)$ by using them.

Remark. For $d \in \{19, 70, 111\}$, the minimal period of the simple continued fraction expansion of $\sqrt{d}$ is 6. The calculations of u_1, u_2, w, v_1, v_2 and z make Table I. We see that all of three cases, $u_1 < u_2$, $u_1 = u_2$ and $u_1 > u_2$, are possible.

Table I

d	u_1	u_2	w	v_1	v_2	z
19	1	1	1	3	0	0
70	2	0	1	2	0	0
111	0	1	0	8	2	0

Theorem 2. *Under the above setting, the value of* $f(s_0)$ *is determined by that of* s_0, $g(s_0)$ *and* $h(s_0)$ *as follows:*

$$(2.5) \qquad s_0 = v_1 - v_2 + (-1)^L + \delta,$$

$$(2.6) \qquad g(s_0) = \frac{1}{q_L} \{\gamma Q + 2(q_{L-1} - r_L)\} + a_L,$$

$$(2.7) \qquad h(s_0) = \frac{1}{q_L} (\gamma - (-1)^L r_L^2 + 1) R + (-1)^L s_0,$$

where

$$\begin{aligned} (2.8)\qquad \gamma &:= q_L(\delta q_L + u_2 - u_1) + w, \\ Q &:= q_{L+1} + q_{L-1} (= a_L q_L + 2q_{L-1}), \\ R &:= r_{L+1} + r_{L-1} (= a_L r_L + 2r_{L-1}). \end{aligned}$$

Proof. First we show that

$$(2.9)\qquad (-1)^\ell \frac{BC}{A} = v_1 - v_2 + (-1)^L + \frac{1}{q_L}\left(u_1 - u_2 - \frac{w+1}{q_L} + \frac{2r_L}{Qq_L}\right).$$

Noting Lemma, we have

$$B = \frac{Ar_L}{q_L} - (-1)^L,$$
$$r_L C - (-1)^L R = (r_L^2 - (-1)^L)R.$$

Hence by using (2.2) and ℓ even, we have

$$\begin{aligned} (2.10)\qquad (-1)^\ell \frac{BC}{A} &= \frac{r_L C}{q_L} - (-1)^L \frac{C}{A} \\ &= \frac{(r_L^2 - (-1)^L)R}{q_L} + (-1)^L\left(\frac{R}{q_L} - \frac{C}{A}\right) \\ &= v_1 + \frac{u_1}{q_L} + (-1)^L\left(\frac{R}{q_L} - \frac{C}{A}\right). \end{aligned}$$

Let E_1 be the last term of the right hand-side of Eq. (2.10). Then we have

$$E_1 = \frac{(-1)^L}{q_L}\left(R - \frac{C}{Q}\right).$$

Moreover, we put

$$E_2 := \frac{a_L + (-1)^L (r_L - q_{L-1})R}{Q}.$$

Here we remark that the relation

$$(2.11)\qquad q_L r_{L-1} - q_{L-1} r_L = (-1)^{L-1}$$

holds by (2.1). Then we have

$$\begin{aligned} (2.12)\quad Qr_{L-1} - C &= (a_L q_L + 2q_{L-1})r_{L-1} - (a_L r_L + 2r_{L-1})r_L \\ &= a_L(q_L r_{L-1} - r_L^2) + 2(q_{L-1} - r_L)r_{L-1} \\ &= a_L(q_{L-1} r_L + (-1)^{L-1} - r_L^2) \\ &\quad + 2(q_{L-1} - r_L)r_{L-1} \\ &= -(-1)^L a_L + a_L(q_{L-1} - r_L)r_L \\ &\quad + 2(q_{L-1} - r_L)r_{L-1} \\ &= -(-1)^L a_L + (q_{L-1} - r_L)(a_L r_L + 2r_{L-1}) \\ &= -(-1)^L a_L - (r_L - q_{L-1})R \\ &= -(-1)^L E_2 Q, \end{aligned}$$

and hence

$$\begin{aligned} (2.13)\qquad E_1 &= \frac{(-1)^L}{q_L}\left(r_{L+1} + r_{L-1} - \frac{C}{Q}\right) \\ &= \frac{(-1)^L}{q_L}\left\{r_{L+1} + \frac{1}{Q}(Qr_{L-1} - C)\right\} \\ &= \frac{(-1)^L r_{L+1}}{q_L} - \frac{E_2}{q_L}. \end{aligned}$$

On the other hand, by (2.11) we have

$$\begin{aligned} (2.14)\quad Qr_L - q_L R &= q_{L+1} r_L + q_{L-1} r_L \\ &\quad - q_L r_{L+1} - q_L r_{L-1} \\ &= (-1)^L - (-1)^{L-1} = (-1)^L 2, \end{aligned}$$

and hence

$$\frac{R}{Q} = \frac{r_L}{q_L} - \frac{(-1)^L 2}{Qq_L}.$$

From this together with (2.3), we have

$$\begin{aligned} (2.15)\quad E_2 &= \frac{a_L}{Q} + (-1)^L (r_L - q_{L-1})\frac{R}{Q} \\ &= \frac{a_L}{Q} + (-1)^L (r_L - q_{L-1})\left(\frac{r_L}{q_L} - \frac{(-1)^L 2}{Qq_L}\right) \\ &= \frac{(-1)^L (r_L - q_{L-1})r_L}{q_L} \\ &\quad + \frac{a_L q_L - 2r_L + 2q_{L-1}}{Qq_L} \\ &= z + \frac{w}{q_L} + \frac{Q - 2r_L}{Qq_L}. \end{aligned}$$

Note that (2.4) gives the equation

$$(2.16)\qquad \frac{(-1)^L r_{L+1} - z}{q_L} = (-1)^L - v_2 - \frac{u_2}{q_L}.$$

Then by (2.13), (2.15) and (2.16), we have

$$\begin{aligned} E_1 &= \frac{(-1)^L r_{L+1} - z}{q_L} - \frac{w}{q_L^2} - \frac{Q - 2r_L}{Qq_L^2} \\ &= (-1)^L - v_2 + \frac{1}{q_L}\left(-u_2 - \frac{w+1}{q_L} + \frac{2r_L}{Qq_L}\right). \end{aligned}$$

Substituting this into (2.10), we get (2.9).

For brevity, we denote by E the last term of the right hand-side of (2.9) and then

$$E = \frac{1}{q_L}\left(u_1 - u_2 - \frac{w+1}{q_L}\right) + \frac{2r_L}{Qq_L^2}.$$

Then by $0 \le w \le q_L - 1$, we have

$$\text{(2.17)} \qquad \frac{1}{q_L} \le \frac{w+1}{q_L} \le 1.$$

Since $Q = a_L q_L + 2q_{L-1} \ge q_L + 2$, we have

$$\text{(2.18)} \qquad Qq_L - 2r_L \ge q_L^2 + 2(q_L - r_L) > 0.$$

When $u_1 < u_2$, it holds from $u_1 \ge 0$ and $u_2 \le q_L - 1$ that $1 - q_L \le u_1 - u_2 \le -1$. Then by (2.17), we have

$$-q_L \le u_1 - u_2 - \frac{w+1}{q_L} \le -1 - \frac{1}{q_L} < -1,$$

and consequently,

$$-1 < E \le -\frac{1}{q_L} + \frac{2r_L}{Qq_L^2} = -\frac{1}{Qq_L^2}(Qq_L - 2r_L).$$

Then by (2.18), we have $-1 < E < 0$, that is, $E + 1$ is the decimal part of $(-1)^\ell BC/A$. Hence by the definition of s_0, we have $s_0 = v_1 - v_2 + (-1)^L$. When $u_1 = u_2$, by (2.17), we have

$$-1 < E \le -\frac{1}{q_L^2} + \frac{2r_L}{Qq_L^2} = -\frac{1}{Qq_L^2}(Q - 2r_L).$$

First, assume that $Q > 2r_L$. Then this inequality yields that $-1 < E < 0$ and we have $s_0 = v_1 - v_2 + (-1)^L$. Next, assume that $Q \le 2r_L$. If we assume $w = 0$ then (2.3) implies that

$$(-1)^L (r_L - q_{L-1}) r_L \equiv 0 \pmod{q_L}.$$

As $\gcd(r_L, q_L) = 1$ by (2.1), we obtain $r_L - q_{L-1} \equiv 0 \pmod{q_L}$. So, $r_L - q_{L-1} = tq_L$ with some integer t. It follows from $r_L \le q_L$ and $0 < q_{L-1}$ that $t \le 0$. Hence,

$$\begin{aligned} Q - 2r_L &= a_L q_L + 2q_{L-1} - 2r_L \\ &> 2(q_{L-1} - r_L) = -2tq_L \ge 0. \end{aligned}$$

This contradicts $Q \le 2r_L$. Thus, $w > 0$ holds. Then by $Q > q_L$, we have

$$\begin{aligned} E &= -\frac{w+1}{q_L^2} + \frac{2r_L}{Qq_L^2} \le -\frac{2}{q_L^2} + \frac{2r_L}{Qq_L^2} \\ &= \frac{2}{Qq_L^2}(r_L - Q) < \frac{2}{Qq_L^2}(r_L - q_L) \le 0. \end{aligned}$$

Hence we have $-1 < E < 0$ so that $s_0 = v_1 - v_2 + (-1)^L$. Finally, when $u_1 > u_2$, it holds from $u_1 \le q_L - 1$ and $u_2 \ge 0$ that $1 \le u_1 - u_2 \le q_L - 1$. Then by (2.17), we have

$$0 \le u_1 - u_2 - \frac{w+1}{q_L} \le q_L - 1 - \frac{1}{q_L} < q_L - 1,$$

and consequently,

$$\begin{aligned} 0 < \frac{2r_L}{Qq_L^2} \le E &< 1 - \frac{1}{q_L} + \frac{2r_L}{Qq_L^2} \\ &= 1 - \frac{1}{Qq_L^2}(Qq_L - 2r_L). \end{aligned}$$

By (2.18), we have $0 < E < 1$, that is, E is the decimal part of $(-1)^\ell BC/A$. Thus we get $s_0 = v_1 - v_2 + (-1)^L + 1$.

Next, let us calculate $g(s_0)$. By using (2.2) and (2.4), we have

$$\text{(2.19)} \qquad \begin{aligned} q_L s_0 &= q_L(v_1 - v_2 + (-1)^L + \delta) \\ &= \{(r_L^2 - (-1)^L)R - u_1\} \\ &\quad - \{(-1)^L(q_L - r_{L+1}) + z - u_2\} \\ &\quad + q_L((-1)^L + \delta) \\ &= r_L^2 R - (-1)^L R + (-1)^L r_{L+1} \\ &\quad - u_1 - z + u_2 + \delta q_L \\ &= r_L C - (-1)^L r_{L-1} + S, \end{aligned}$$

where we put $S := \delta q_L + u_2 - u_1 - z$. Then we have

$$\begin{aligned} As_0 &= Qq_L s_0 \\ &= Qr_L C - (-1)^L Qr_{L-1} + QS \\ &= (B + (-1)^L)C - (-1)^L Qr_{L-1} + QS \\ &= BC + (-1)^L(C - Qr_{L-1}) + QS. \end{aligned}$$

Hence by (2.12), we have

$$\text{(2.20)} \qquad \begin{aligned} g(s_0) &= As_0 - BC \\ &= (-1)^L(C - Qr_{L-1}) + QS \\ &= E_2 Q + QS \\ &= a_L + (-1)^L(r_L - q_{L-1})R + QS. \end{aligned}$$

Here, it follows from (2.3) and (2.8) that

$$\text{(2.21)} \quad \begin{aligned} S &= \delta q_L + u_2 - u_1 - z \\ &= \delta q_L + u_2 - u_1 - \frac{(-1)^L(r_L - q_{L-1})r_L - w}{q_L} \\ &= \frac{1}{q_L}\{q_L(\delta q_L + u_2 - u_1) + w \\ &\quad - (-1)^L(r_L - q_{L-1})r_L\} \\ &= \frac{1}{q_L}\{\gamma - (-1)^L(r_L - q_{L-1})r_L\}. \end{aligned}$$

By substituting this into (2.20) and by using (2.14), therefore, we get

$$\begin{aligned}
g(s_0) &= a_L + (-1)^L(r_L - q_{L-1})R \\
&\quad + \frac{Q}{q_L}\{\gamma - (-1)^L(r_L - q_{L-1})r_L\} \\
&= a_L + \frac{1}{q_L}\{(-1)^L(r_L - q_{L-1})q_L R \\
&\quad + Q\gamma - (-1)^L(r_L - q_{L-1})Qr_L\} \\
&= a_L + \frac{1}{q_L}\{Q\gamma + (-1)^L(r_L - q_{L-1})(q_L R - Qr_L)\} \\
&= a_L + \frac{1}{q_L}\{Q\gamma + 2(q_{L-1} - r_L)\}.
\end{aligned}$$

Next let us calculate $h(s_0)$. By (2.14), we have

$$B = Qr_L - (-1)^L = q_L R + (-1)^L.$$

Then by (2.19), we have

$$\begin{aligned}
Bs_0 &= q_L R s_0 + (-1)^L s_0 \\
&= Rr_L C - (-1)^L Rr_{L-1} \\
&\quad + R(\delta q_L + u_2 - u_1 - z) + (-1)^L s_0 \\
&= C^2 - (-1)^L Rr_{L-1} \\
&\quad + R(\delta q_L + u_2 - u_1 - z) + (-1)^L s_0.
\end{aligned}$$

By (2.21) and (2.11), therefore, we have

$$\begin{aligned}
h(s_0) &= Bs_0 - C^2 \\
&= -(-1)^L Rr_{L-1} + R(\delta q_L + u_2 - u_1 - z) + (-1)^L s_0 \\
&= -(-1)^L Rr_{L-1} \\
&\quad + \frac{R}{q_L}\{\gamma - (-1)^L(r_L - q_{L-1})r_L\} + (-1)^L s_0 \\
&= \frac{R}{q_L}\{\gamma - (-1)^L(r_L^2 + q_L r_{L-1} - q_{L-1}r_L)\} + (-1)^L s_0 \\
&= \frac{R}{q_L}\{\gamma - (-1)^L(r_L^2 + (-1)^{L-1})\} + (-1)^L s_0 \\
&= \frac{R}{q_L}(\gamma - (-1)^L r_L^2 + 1) + (-1)^L s_0.
\end{aligned}$$

Theorem 2 is now proved. □

Example 1. Let $L = 2$ and consider a symmetric string a_1, a_2, a_1. Then we have Table II. Hence by

$$(r_L^2 - (-1)^L)(r_{L+1} + r_{L-1}) = 0 = 0q_L + 0,$$
$$(-1)^L(r_L - q_{L-1})r_L = 0 = 0q_L + 0,$$

we obtain $u_1 = 0, v_1 = 0, w = 0, z = 0$. Moreover, we have

$$(-1)^L(q_L - r_{L+1}) + z = (-1)^2(a_1 - a_2) = a_1 - a_2.$$

Table II

n	0	1	2	3
q_n	0	1	a_1	$a_1a_2 + 1$
r_n	1	0	1	a_2

First we consider the case where $a_1 \mid a_2$. Since $q_L = a_1$ and

$$(-1)^2(q_L - r_{L+1}) + z = 0 + a_1(1 - a_2/a_1),$$

we have $u_2 = 0$, $v_2 = 1 - a_2/a_1$. Then we have $u_1 = 0 = u_2$, so $\delta = 0$. Hence by (2.5), we have

$$s_0 = v_1 - v_2 + (-1)^L = a_2/a_1.$$

Since

$$\gamma = q_L(\delta q_L + u_2 - u_1) + w = 0,$$

it follows from (2.6), (2.7) that $g(s_0) = a_2$ and $h(s_0) = a_2/a_1$.

Next, we consider the case where $a_1 \nmid a_2$. Let r be the remainder of the division of a_2 by a_1, that is,

$$a_2 = a_1[a_2/a_1] + r, \quad 0 < r < a_1.$$

Then we have

$$(-1)^L(q_L - r_{L+1}) + z = a_1 - r + a_1(-[a_2/a_1]),$$

and hence $u_2 = a_1 - r$, $v_2 = -[a_2/a_1]$. Then we have $u_1 = 0 < a_1 - r = u_2$, and hence $\delta = 0$. By (2.5), therefore, we have

$$s_0 = v_1 - v_2 + (-1)^L = [a_2/a_1] + 1.$$

Since

$$\gamma = q_L(\delta q_L + u_2 - u_1) + w = a_1(a_1 - r),$$

it follows from (2.6), (2.7) that

$$g(s_0) = (a_1 - r)(a_1a_2 + 2) + a_2,$$
$$h(s_0) = (a_1 - r)a_2 + [a_2/a_1] + 1.$$

Example 2. Let $L = 4$, t be a positive integer, and consider a symmetric string $2, 2, 1, 7t, 1, 2, 2$. Then we have Table III. Hence by

$$(r_L^2 - (-1)^L)(r_{L+1} + r_{L-1}) = 7(24t + 4) + 4,$$
$$(-1)^L(r_L - q_{L-1})r_L = -6 = 7(-1) + 1,$$

we obtain $u_1 = 4$, $v_1 = 24t + 4$, $w = 1$, $z = -1$. Since $(-1)^L(q_L - r_{L+1}) + z = 7(-3t) + 4$, we have $u_2 = 4$, $v_2 = -3t$. This yields that $\delta = 0$ and $\gamma = 1$. Hence we see by Theorem 2 that

$$s_0 = 27t + 5, \quad g(s_0) = 2(7t + 1), \quad h(s_0) = 6t + 1,$$

Table III

n	0	1	2	3	4	5
q_n	0	1	2	5	7	$49t+5$
r_n	1	0	1	2	3	$21t+2$

so that

$$d(t) := f(s_0)/4 = 7^2t^2 + 4\cdot 5t + 2.$$

Since $A = 7(49t+10)$ and $C = 3(21t+4)$ by Lemma, if t is even (resp. odd) then Case (II) occurs (resp. Case (I) occurs and s_0 is even). The assumption (1.2) of Theorem 1 holds: $g(s_0) > 2, 7t$. Hence, Theorem 1 implies that the simple continued fraction expansion of $\sqrt{d(t)}$ is

$$\sqrt{d(t)} = [7t+1, \overline{2, 2, 1, 7t, 1, 2, 2, 14t+2}]$$

and $d(t)$ is a positive integer with period 8 of minimal type for $\sqrt{d(t)}$. The discriminant $D(d(t))$ of $d(t)$ is $8\,(\neq 0)$. We see by a result of Nagell (cf. [3, Proposition 6.1]) that the set $\{d(2u) \mid u \in \mathbf{N}\}$ (resp. $\{d(2u-1) \mid u \in \mathbf{N}\}$) contains infinite square-free elements. Consequently, we can choose a sequence $\{d_n\}_{n\geq 1}$ of square-free positive integers which is strictly monotonously increasing such that $d_n \equiv 2 \pmod 4$ (resp. $d_n \equiv 3 \pmod 4$). Thus there exist infinitely many real quadratic fields with period 8 of minimal type. (We see that $d(1) = 71$, $d(3) = 503$ and the class numbers of real quadratic fields $\mathbf{Q}(\sqrt{71})$ and $\mathbf{Q}(\sqrt{503})$ are both equal to 1.)

Finally, we give the factorization of $f(x)$. We will use it to construct systematically such fields as in Example 2 for any even period (≥ 6).

Proposition. *Put $\ell = 2L$. Then $f(x)$ can be factored into two polynomials of degree* 1:

$$f(x) = \{q_L^2x - r_L^2(q_LR - (-1)^L)\} \times \{Q^2x - R^2(Qr_L + (-1)^L)\},$$

where $Q := q_{L+1} + q_{L-1}$, $R := r_{L+1} + r_{L-1}$.

Proof. By the definition of $f(x)$, we have

$$\begin{aligned} f(x) &= (Ax - BC)^2 + 4(Bx - C^2) \\ &= A^2x^2 - 2(AC-2)Bx + (B^2-4)C^2. \end{aligned}$$

On the other hand, since it follows from (2.14) and Lemma that

$$\begin{aligned} q_LR - (-1)^L &= Qr_L - (-1)^L3 \\ &= B + (-1)^L - (-1)^L3 \\ &= B - (-1)^L2, \\ Qr_L + (-1)^L &= B + (-1)^L2, \end{aligned}$$

we obtain

$$\begin{aligned} &\{q_L^2x - r_L^2(q_LR - (-1)^L)\} \\ &\quad \times \{Q^2x - R^2(Qr_L + (-1)^L)\} \\ &= q_L^2Q^2x^2 \\ &\quad - \{q_L^2R^2(B + (-1)^L2) + Q^2r_L^2(B - (-1)^L2)\}x \\ &\quad + r_L^2R^2(B - (-1)^L2)(B + (-1)^L2) \\ &= A^2x^2 \\ &\quad - \{q_L^2R^2(B + (-1)^L2) + Q^2r_L^2(B - (-1)^L2)\}x \\ &\quad + C^2(B^2 - 4). \end{aligned}$$

Here, as $B^2 = AC + 1$ by Lemma, we have

$$\begin{aligned} &q_L^2R^2(B + (-1)^L2) + Q^2r_L^2(B - (-1)^L2) \\ &= (B - (-1)^L)^2(B + (-1)^L2) \\ &\quad + (B + (-1)^L)^2(B - (-1)^L2) \\ &= (B^2 - (-1)^L2B + 1)(B + (-1)^L2) \\ &\quad + (B^2 + (-1)^L2B + 1)(B - (-1)^L2) \\ &= (2B^2 + 2)B + (-(-1)^L4B)(-1)^L2 \\ &= 2(B^2 - 3)B = 2(AC - 2)B. \end{aligned}$$

The proof is completed. □

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