Differential Harnack inequality for the nonlinear heat equations

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Abstract: In this paper, we establish some differential Harnack inequalities for positive solutions to the nonlinear heat equations with potentials evolving by the Bernhard List's flow. Our theorems generalize Cao and Zhang's results [1].

Key words: Differential Harnack inequality; Bernhard List's flow; gradient estimate.

1. Introduction. The differential Harnack estimates for parabolic equations have become one of important tools in the study of geometric analysis. The work in the aspect originated in Li and Yau's paper [6], in which they proved a differential Harnack inequality for positive solutions to the heat equation on Riemannian manifolds with a fixed metric. Later, Yau [9] generalized this result to Harnack inequalities for some nonlinear heat-type equation. Since then, Harnack estimates for positive solutions to the heat equation coupled with geometric flows have been widely studied (see [2, 3, 5, 8]). Recently, Cao and Zhang [1] proved an interesting differential Harnack inequality for positive solutions to the forward nonlinear heat equation

$$\frac{\partial f}{\partial t} = \triangle f - f \ln f + Rf \tag{1.1}$$

coupled with the Ricci flow. They got the following result.

Theorem 1.1 (Cao and Zhang [1])**.** *Let $(M, g(t)), t \in [0, T)$, be a solution to the Ricci flow on a closed manifold, and suppose that $g(0)$ has weakly positive curvature operator. Let f be a positive solution to the heat equation* (1.1)*, $u = -\ln f$ and*

$$H = 2\triangle u - |\nabla u|^2 - 3R - \frac{2n}{t}.$$

Then for $\forall t \in (0, T)$,

$$H \leq \frac{n}{4},$$

here $(0, T)$ means some time interval.

If the equation (1.1) is changed into

$$\frac{\partial f}{\partial t} = \triangle f - \frac{f \ln f}{1 + \frac{t}{2}} + Rf, \tag{1.2}$$

under the same assumption as Theorem 1.1, the authors [1] obtained

$$H \leq 0.$$

Very recently, Fang [4] studied the linear heat equations with potentials on closed Riemannian manifolds evolving by the Bernhard List's flow

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2R_{ij} + 2\alpha_n d\psi \otimes d\psi, \\ \dfrac{\partial \psi}{\partial t} = \triangle_{g(t)} \psi, \end{cases}$$

where $\psi : M \to R$ is a smooth function and $\alpha_n = \frac{n-1}{n-2}$. Let f be a positive solution of the time-dependent nonlinear heat equation with potential, i.e.,

$$\frac{\partial f}{\partial t} = \triangle f + c(R - \alpha_n |\nabla \psi|^2) f, \tag{1.3}$$

where c is any constant. Under the Bernhard List's flow, we have $\frac{d}{dt}\int_M f dv = 0$. The flow was first introduced by Bernhard List [7] and Fang [4] proved some differential Harnack estimate for positive solutions to the linear heat equations with potentials under this flow. The author in [4] introduced a symmetric two-tensor on $(M, g(t))$ with components $S_{ij} = R_{ij} - \alpha_n \nabla_i \psi \nabla_j \psi$ and its trace $S = g^{ij} S_{ij} = R - \alpha_n |\nabla \psi|^2$, moreover, he defined a differential Harnack type quantity

$$\mathcal{H}(\mathcal{S}, X) = \frac{\partial S}{\partial t} + 2\langle \nabla S, X\rangle + 2\mathcal{S}(X, X) + \frac{S}{t}.$$

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Under these symbols, the author [4] gave the following result.

Theorem 1.2 (Fang [4]). *Let* $(g(t), \psi(t)), t \in [0, T)$*, be a solution to the Bernhard List's flow on a closed manifold M, and suppose that* $\mathcal{H}(\mathcal{S}, X)$ *is nonnegative for* $\forall X \in \Gamma(TM)$ *and all times* $t \in [0, T)$*. Let* f *be a positive solution to the heat equation* (1.3)*,* $u = -lnf$*, and*

$$P = 2\triangle u - |\nabla u|^2 - 3S - \frac{2n}{t}.$$

Then for $\forall t \in (0, T)$

$$P \leq 0.$$

In this paper, we will consider the nonlinear heat equation

$$\frac{\partial f}{\partial t} = \triangle f - flnf + (R - \alpha_n |\nabla \psi|^2) f \tag{1.4}$$

coupled with Bernhard List's flow. Now, we give our first main result.

Theorem 1.3. *Let* $(g(t), \psi(t)), t \in [0, T)$*, be a solution to the Bernhard List's flow on a closed manifold M. Suppose* $\mathcal{H}(\mathcal{S}, X)$ *is nonnegative for* $\forall X \in \Gamma(TM)$ *and all times* $t \in [0, T)$*. Let* f *be a positive solution to the heat equation* (1.4)*,* $u = -lnf$*, and*

$$H = 2\triangle u - |\nabla u|^2 - 3S - \frac{2n}{t}.$$

Then for $\forall t \in (0, T)$

$$H \leq \frac{n}{4}.$$

Secondly, we will also consider the positive solutions to the nonlinear heat equation

$$\frac{\partial f}{\partial t} = \triangle f - \frac{flnf}{1 + \frac{t}{2}} + (R - \alpha_n |\nabla \psi|^2) f. \tag{1.5}$$

We can get our second main result.

Theorem 1.4. *Let* $(g(t), \psi(t)), t \in [0, T)$*, be a solution to the Bernhard List's flow on a closed manifold M. Suppose* $\mathcal{H}(\mathcal{S}, X)$ *is nonnegative for* $\forall X \in \Gamma(TM)$ *and all times* $t \in [0, T)$*. Let* f *be a positive solution to the heat equation* (1.5)*,* $u = -lnf$*, and*

$$P = 2\triangle u - |\nabla u|^2 - 3S - \frac{2n}{t}.$$

Then for $\forall t \in (0, T)$

$$P \leq 0.$$

2. Proof of Theorem 1.2. Let us consider positive solutions to equation (1.4), assume $f = e^{-u}$, then we have

$$\frac{\partial u}{\partial t} = \triangle u - |\nabla u|^2 - u - S. \tag{2.1}$$

Proof. From the definition of H in Theorem 1.2 and comparing with [4, Lemma 2.2], we have

$$\begin{aligned} \frac{\partial}{\partial t}(\triangle u) &= \triangle(\triangle u) - \triangle(|\nabla u|^2) - \triangle S + 2S_{ij}u_{ij} \\ &\quad - 2\alpha_n \triangle\psi \nabla\psi \cdot \nabla u - \triangle u, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial t}(|\nabla u|^2) &= 2\nabla(\triangle u) \cdot \nabla u - 2\nabla(|\nabla u|^2) \cdot \nabla u \\ &\quad - 2|\nabla u|^2 - 2\nabla u \cdot \nabla S + 2S_{ij}u_i u_j \\ &= \triangle(|\nabla u|^2) - 2|\nabla\nabla u|^2 - 2\nabla(|\nabla u|^2) \cdot \nabla u \\ &\quad - 2|\nabla u|^2 - 2\nabla u \cdot \nabla S - 2\alpha_n(\nabla\psi \cdot \nabla u)^2. \end{aligned}$$

Let

$$H = 2\triangle u - |\nabla u|^2 - 3S - \frac{2n}{t}.$$

Comparing with [4, Corollary 2.1], we have

$$\begin{aligned} \frac{\partial H}{\partial t} &= \triangle H - 2\nabla H \cdot \nabla u - 2\left|u_{ij} - S_{ij} - \frac{1}{t}g_{ij}\right|^2 \\ &\quad - \frac{2}{t}H - 2\alpha_n(\triangle\psi + \nabla\psi \cdot \nabla u)^2 \\ &\quad - \frac{2}{t}|\nabla u|^2 - 2\mathcal{H}(\mathcal{S}, \nabla u) - 2\triangle u + 2|\nabla u|^2, \end{aligned}$$

where the last terms of the right hand side coming from the extra term $-u$ in (2.1). Since

$$-2\triangle u + 2|\nabla u|^2 = -H + |\nabla u|^2 - 3S - \frac{2n}{t},$$

we have

$$\begin{aligned} \frac{\partial H}{\partial t} &= \triangle H - 2\nabla H \cdot \nabla u - 2\left|u_{ij} - S_{ij} - \frac{1}{t}g_{ij}\right|^2 \\ &\quad - \left(\frac{2}{t} + 1\right)H + \left(1 - \frac{2}{t}\right)|\nabla u|^2 - 3S \\ &\quad - \frac{2n}{t} - 2\alpha_n(\triangle\psi + \nabla\psi \cdot \nabla u)^2 \\ &\quad - \frac{2}{t}|\nabla u|^2 - 2\mathcal{H}(\mathcal{S}, \nabla u). \end{aligned}$$

From the definition of H, we know

$$|\nabla u|^2 = 2\left(\triangle u - S - \frac{t}{n}\right) - H - S.$$

Now we can compute the evolution equation of H as follows:

$$\begin{aligned}
\frac{\partial H}{\partial t} &\le \triangle H - 2\nabla H\cdot\nabla u - 2\left|u_{ij} - S_{ij} - \frac{1}{t}g_{ij}\right|^2 \\
&\quad + 2\left(\triangle u - S - \frac{n}{t}\right) - \left(\frac{2}{t}+1\right)H \\
&\quad - \frac{2}{t}|\nabla u|^2 - 4S - H - \frac{2n}{t} \\
&\quad - 2\alpha_n(\triangle\psi + \nabla\psi\cdot\nabla u)^2 - \frac{2}{t}|\nabla u|^2 \\
&\quad - 2\mathcal{H}(\mathcal{S},\nabla u) \\
&\le \triangle H - 2\nabla H\cdot\nabla u - \frac{2}{n}\left(\triangle u - S - \frac{n}{t}\right)^2 \\
&\quad + 2\left(\triangle u - S - \frac{n}{t}\right) - \left(\frac{2}{t}+1\right)H \\
&\quad - \frac{2}{t}|\nabla u|^2 - 4S - H - \frac{2n}{t} \\
&\quad - 2\alpha_n(\triangle\psi + \nabla\psi\cdot\nabla u)^2 - 2\mathcal{H}(\mathcal{S},\nabla u) \\
&= \triangle H - 2\nabla H\cdot\nabla u - \left(\frac{2}{t}+2\right)H - \frac{2}{t}|\nabla u|^2 \\
&\quad - 4S - \frac{2n}{t} - \frac{2}{n}\left(\triangle u - S - \frac{n}{t} - \frac{n}{2}\right)^2 \\
&\quad - 2\alpha_n(\triangle\psi + \nabla\psi\cdot\nabla u)^2 + \frac{n}{2},
\end{aligned}$$

where in the above second inequality, we used the elementary inequality

$$\left|u_{ij} - S_{ij} - \frac{1}{t}g_{ij}\right|^2 \ge \frac{1}{n}\left(\triangle u - S - \frac{n}{t}\right)^2.$$

Note that the evolution equation of S under Bernhard List's flow is

$$\frac{\partial}{\partial t}S = \triangle S + 2|S_{ij}|^2 + 2\alpha_n(\triangle\psi)^2 \ge \triangle S + \frac{2}{n}S^2.$$

Applying the maximum principle to this inequality yields

$$S \ge -\frac{n}{2t}. \tag{2.2}$$

Adding $-\frac{n}{4}$ to H, we have

$$\begin{aligned}
\frac{\partial(H-\frac{n}{4})}{\partial t} &\le \triangle\left(H - \frac{n}{4}\right) - 2\nabla\left(H - \frac{n}{4}\right)\cdot\nabla u \\
&\quad - \left(\frac{2}{t}+2\right)\left(H - \frac{n}{4}\right) - \frac{2}{t}|\nabla u|^2 \\
&\quad - \frac{2}{n}\left(\triangle u - S - \frac{n}{t} - \frac{n}{2}\right)^2 \\
&\quad - 2\alpha_n(\triangle\psi + \nabla\psi\cdot\nabla u)^2 - \frac{n}{2t}.
\end{aligned}$$

It is easy to see that for t is small enough, $H - \frac{n}{4} < 0$, then by the maximum principle, we obtain

$$H \le \frac{n}{4}.$$

We complete the proof. □

As a consequence of Theorem 1.2, we have the following integral Harnack inequality. We refer the reader to [1] for analogous details of proof.

Corollary 2.1. *Let $(g(t),\psi(t)), t\in[0,T)$, be a solution to the Bernhard List's flow on a closed manifold M and f be positive solutions to nonlinear equation* (1.3)*. Suppose that $\mathcal{H}(\mathcal{S},X)$ is nonnegative for $\forall X\in\Gamma(TM)$ and all times $t\in[0,T)$. Let (x_1,t_2) and (x_2,t_2), $0<t_1<t_2$, be two points in $M\times(0,T)$, and $\Gamma = \inf_{\gamma}\int_{t_1}^{t_2} e^t(|\dot\gamma|^2 + S + \frac{2n}{t} + \frac{n}{4})dt$, where γ is any space time path joining (x_1,t_2) and (x_2,t_2). Then we have*

$$-e^{t_2}lnf(x_2,t_2) + e^{t_1}lnf(x_1,t_1) \le \frac{1}{2}\Gamma.$$

3. Proof of Theorem 1.3. In this section we study u satisfying the equation (1.5) coupled with Bernhard flow, we investigate the same quantity

$$H = 2\triangle u - |\nabla u|^2 - 3S - \frac{2n}{t}.$$

Proof. Direct computation gives the following evolution equation.

$$\begin{aligned}
\frac{\partial H}{\partial t} &= \triangle H - 2\nabla H\cdot\nabla u - 2\left|u_{ij} - S_{ij} - \frac{1}{t}g_{ij}\right|^2 \\
&\quad - \frac{2}{t}H - 2\alpha_n(\triangle\psi + \nabla\psi\cdot\nabla u)^2 \\
&\quad - \frac{2}{t}|\nabla u|^2 - 2\mathcal{H}(\mathcal{S},\nabla u) \\
&\quad + \frac{2}{t+2}(-2\triangle u + 2|\nabla u|^2),
\end{aligned}$$

where the last terms of the right hand side coming from the extra term $\frac{-u}{1+\frac{t}{2}}$ in (2.1). Since

$$-2\triangle u+2|\nabla u|^2=-H+|\nabla u|^2-3S-\frac{2n}{t},$$

we have

$$\begin{aligned}\frac{\partial H}{\partial t}&=\triangle H-2\nabla H\cdot\nabla u-2\left|u_{ij}-S_{ij}-\frac{1}{t}g_{ij}\right|^2\\&\quad-\left(\frac{2}{t}+\frac{2}{t+2}\right)H+\left(\frac{2}{t+2}-\frac{2}{t}\right)|\nabla u|^2\\&\quad-\frac{4n}{t^2+2t}-2\alpha_n(\triangle\psi+\nabla\psi\cdot\nabla u)^2\\&\quad-\frac{6}{t+2}S-2\mathcal{H}(\mathcal{S},\nabla u).\end{aligned}$$

By the inequality (2.2), we have

$$\begin{aligned}\frac{\partial H}{\partial t}&\le\triangle H-2\nabla H\cdot\nabla u-2\left|u_{ij}-S_{ij}-\frac{1}{t}g_{ij}\right|^2\\&\quad-\left(\frac{2}{t}+\frac{2}{t+2}\right)H+\left(\frac{2}{t+2}-\frac{2}{t}\right)|\nabla u|^2\\&\quad-\frac{n}{t^2+2t}-2\alpha_n(\triangle\psi+\nabla\psi\cdot\nabla u)^2\\&\quad-2\mathcal{H}(\mathcal{S},\nabla u).\end{aligned}$$

It is easy to see that for t is small enough, $H<0$, then by the maximum principle, we obtain

$$H\le 0.$$

□

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