Simplicity of the lowest eigenvalue of non-commutative harmonic oscillators and the Riemann scheme of a certain Heun's differential equation

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Abstract: The non-commutative harmonic oscillator (NcHO) is a special type of self-adjoint ordinary differential operator with non-commutative coefficients. In the present note, we aim to provide a reasonable criterion that derives the simplicity of the lowest eigenvalue of NcHO. It actually proves the simplicity of the lowest eigenvalue for a large class of structure parameters. Moreover, this note describes a certain equivalence between the spectral problem of the NcHO (for the even parity) and existence of holomorphic solutions of Heun's ordinary differential equations in a complex domain. The corresponding Riemann scheme allows us to give another proof to the criterion.

Key words: Non-commutative harmonic oscillators; lowest eigenvalue; multiplicity of eigenvalues; oscillator representation; Heun's differential equation; Riemann's scheme.

1. Introduction. Let Q be a second order parity-preserving ordinary differential operator defined by

$$Q = A\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\right) + B\left(x\frac{d}{dx} + \frac{1}{2}\right),$$

where $A, B \in \mathrm{Mat}_2(\mathbf{R})$, A is positive definite symmetric, and B is skew-symmetric. We further assume that the Hermitian matrix $A + iB$ is positive definite, that is, $\det(A) > \mathrm{pf}(B)^2$. The former requirement arises from the formal self-adjointness of the operator Q relative to the natural inner product on $L^2(\mathbf{R}, \mathbf{C}^2)(= \mathbf{C}^2 \otimes L^2(\mathbf{R}))$. The latter guarantees that the eigenvalues of Q are all positive and form a discrete set with finite multiplicity. We call Q the *non-commutative harmonic oscillator* (NcHO) [19,20] (see also [17]). It is shown in [20] that one may always assume that Q has the following normal form:

$$\begin{aligned} Q &= Q_{(\alpha,\beta)} \\ &= \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\right) \\ &\quad + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left(x\frac{d}{dx} + \frac{1}{2}\right). \end{aligned}$$

One immediately observes the required conditions above can be described as $\alpha, \beta > 0$ and $\alpha\beta > 1$.

It should be noted that, when $\alpha = \beta$, Q is unitarily equivalent to a couple of quantum harmonic oscillators, whence the eigenvalues are easily calculated as $\{\sqrt{\alpha^2 - 1}(n + \frac{1}{2}) \mid n \in \mathbf{Z}_{\geq 0}\}$ having multiplicity 2 ([20]). Actually, when $\alpha = \beta$, behind Q, there exists a structure corresponding to the tensor product of the 2-dimensional trivial representation and the oscillator representation of the Lie algebra $\mathfrak{sl}_2$ ([5]). This unitarily equivalence is obtained also by a probabilistic construction of the heat semigroup and kernel associated with Q ([25]). The clarification of the spectrum in the general $\alpha \neq \beta$ case is, however, considered to be highly non-trivial. Indeed, while the spectrum is well described theoretically by using certain continued fractions (see [19–22]) and also by Fuchsian ordinary differential equations with four regular singular points in a complex domain (see [12]), and in particular there are some results related to the estimate of upper bound of the lowest eigenvalue [4,7,11,14–16], almost no information is available in reality for the low-lying eigenvalues when $\alpha \neq \beta$. Therefore, in spite of many studies, the spectral description of the NcHO is still incomplete.

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The eigenvalues of NcHO build a continuous curve with arguments α and β (see [11]). It comes as an important problem to analyze the behavior of eigenvalue curves, in particular, the characterization of crossing/avoided crossing of eigenvalue curves for various operators (see e.g. [2,24] and a numerical investigation [11] for the NcHO). Especially, it has been an important problem to determine the structure of the ground state (i.e. the eigenspace of the lowest eigenvalue) of Q when $\alpha \neq \beta$:

i) What is the multiplicity of the lowest eigenvalue?

ii) Does any odd function contribute to the ground state?

In recent years, special attention has been paid to studying the spectrum of self-adjoint operators with non-commutative coefficients, like the Rabi model, the Jaynes-Cumming model, etc., not only in mathematics [2,3] but also in experimental physics (see e.g. [1]). Our NcHO may provide one of these Hamiltonians describing such an interaction between a one-mode photon and a two-level atom. In addition to this direction, in order to get a better understanding to the spectrum (e.g. Weyl's law for the distribution of eigenvalues) and for its own interest (e.g. certain modular properties of Apéry-like numbers [27]), the spectral zeta function for Q has been studied extensively from a number theoretic point of view (see e.g. [6–10]). One notices furthermore that the case $\alpha\beta \leq 1$ is to be explored (see [18] when $\alpha\beta = 1$).

In [20–22], we have constructed the eigenfunctions and eigenvalues in terms of continued fractions determined by a certain three terms recurrence relation, which can be derived from the expansion of eigenfunctions relative to a basis constructed by suitably twisting the classical Hermite functions. We call the eigenfunction $u(x)$ in $L^2(\mathbf{R}, \mathbf{C}^2)$ is of finite type if $u(x)$ can be expanded by a finite number of elements of this Hermite basis. The eigenvalue corresponding to the finite-type eigenfunction is said to be of finite type. Otherwise, we call the eigenvalues/eigenfunction of infinite type. We denote Σ_0 (reps. Σ_∞) the set of eigenvalues corresponding to eigenfunctions of finite (resp. infinite) type. Since the operator Q preserves the parity we define $\Sigma^\pm$ to be the set of eigenvalues whose eigenfunctions are even/odd, that is those satisfying $u(-x) = \pm u(x)$. Then there is a classification of eigenvalues: $\Sigma_0^\pm = \Sigma_0 \cap \Sigma^\pm$ corresponding to even/odd eigenfunctions of finite type and $\Sigma_\infty^\pm = \Sigma_\infty \cap \Sigma^\pm$.

To state our first result we recall ([21]) that

$$\Sigma_0^\pm \subset \Sigma_\infty^\pm \quad \text{and} \quad \Sigma_0^+ \cap \Sigma_0^- = \emptyset. \tag{1}$$

The former implies that $\Sigma_\infty^\pm = \Sigma^\pm$. Moreover, since the dimension of the eigenspace of $\Sigma_\infty^\pm$ is at most 2 ([21]), one notices in particular that the multiplicity of each eigenvalue is at most 3.

We show the following criterion of simplicity of the lowest eigenvalue of NcHO.

Theorem 1.1. *Suppose that the ground state, i.e. the eigenspace of the lowest eigenvalue E of Q, consists of even-functions. Then E is simple.*

The proof of this theorem is rather simple while it uses many basic results obtained so far.

The second concern of this note is to announce a part of the results in [26] giving a complex analytic description of the even eigenspaces of Q as well as the odd case studied in [12]. In harmonic analysis on the real line, in general even/odd eigenspaces have completely analogues structures. Moreover, we could not see any difference between the even/odd eigenspaces in the study [19–22]. However, in the complex domain picture drawn in [12] the odd part Σ^- corresponds to the second order equation given by Heun's ordinary differential equation [23] while the even part Σ^+ corresponds to the third-order equation (constructed by Heun's operator). Therefore resolution of this asymmetry is desirable. In this note we provide the completely parallel structure of even part with that of the odd part. The details of the proof and related subjects are given in [26] and to be published elsewhere. Also, using the picture given by the complex domain description, we will give a second proof of Theorem 1.1.

2. Simplicity of the lowest eigenvalue. We first give a proof of Theorem 1.1. By assumption, it follows from (1) that the multiplicity of E is at most 2. Suppose the multiplicity of E is 2. Then, by the assumption, one observes $E \in \Sigma_0^+$. Hence, by Theorem 1.1 in [20], there exists a nonnegative integer n such that E can be expressed as

$$E = 2\,\frac{\sqrt{\alpha\beta}\sqrt{\alpha\beta-1}}{\alpha+\beta}\left(n+\frac{1}{2}\right).$$

It follows that $E \geq \frac{\sqrt{\alpha\beta}\sqrt{\alpha\beta-1}}{\alpha+\beta}$.

On the other hand, the following important upper bound estimate of E is known (Theorem 8.2.1 in [17]):

$$E \leq \frac{\sqrt{\alpha\beta}\sqrt{\alpha\beta-1}}{\alpha+\beta+|\alpha-\beta|\frac{(\alpha\beta-1)^{1/4}}{\sqrt{\alpha\beta}}\operatorname{Re}\omega}, \tag{2}$$

where $\omega \in \mathbf{C}$ is the solution of $\omega^2 = \sqrt{\alpha\beta-1}-i$ with $\operatorname{Re}\omega > 0$. Thus, in particular, since $\alpha \neq \beta$ we have

$$E < \frac{\sqrt{\alpha\beta}\sqrt{\alpha\beta-1}}{\alpha+\beta}.$$

This contradicts the previous estimate. Hence the theorem follows.

In [4], in the proof of the main result (Theorem 1.1), it is shown that

Lemma 2.1. *Let $\beta > \alpha > \sqrt{2}$. Then there is no odd eigenfunction corresponding the lowest eigenvalue E.*

By this lemma the following result is immediate.

Corollary 2.2. *Suppose $\alpha, \beta > \sqrt{2}$ $(\alpha \neq \beta)$. Then the lowest eigenvalue E is simple and $E \in \Sigma_\infty^+ \backslash \Sigma_0^+$.*

3. Even spectrum of the NcHO and Heun's ODE. In studying the ground state of NcHO, the description of the even eigenfunctions is important. This section describes the results in [26] without proof.

Let H, E and F be an $\mathfrak{sl}_2$-triple:

$$[H, E] = 2E,\ [H, F] = -2F,\ [E, F] = H.$$

Let π denote a (oscillator) representation of $\mathfrak{sl}_2$ on $\mathbf{C}[y]$ given by

$$\pi(H) = y\partial_y + 1/2,\ \pi(E) = y^2/2,\ \pi(F) = -\partial_y^2/2,$$

where $\partial_y = d/dy$. We also introduce another realization of the $\mathfrak{sl}_2$-triple as

$$\varpi(H) = z\partial_z + 1/2,$$
$$\varpi(E) = z^2(z\partial_z+1)/2,\ \varpi(F) = -(2z)^{-1}\partial_z.$$

Similarly to the discussion in [12] we observe the following correspondence between two representations of $\mathfrak{sl}_2$ above (see [26]).

Proposition 3.1. *Let $a \geq 1$. Define a modified Laplace transform $\mathcal{L}_a$ by*

$$(\mathcal{L}_a u)(z) = \int_0^\infty u(yz)e^{-\frac{y^2}{2}}y^{a-1}dy.$$

Then $\mathcal{L}_1$ possesses the following quasi-intertwining property:

$$\mathcal{L}_1\pi(H) = \varpi(H)\mathcal{L}_1,\ \mathcal{L}_1\pi(E) = \varpi(E)\mathcal{L}_1,$$
$$(\mathcal{L}_1\pi(F)u)(z) = \varpi(F)(\mathcal{L}_1 u)(z) + u'(0)/(2z).$$

Let $\mathbf{C}[z, z^{-1}]$ be the set of all Laurent polynomials in z. Since $u'(0) = 0$ for an even polynomial $u(y)$, the equivalence $(\pi, \mathbf{C}[y^2]) \cong (\varpi, \mathbf{C}[z^2])$ as representations of $\mathfrak{sl}_2$ is obvious. Moreover, by the quasi-intertwiner $\mathcal{L}_1$, we obtain the following equivalence between the odd part of the (oscillator) representation $(\pi, y\mathbf{C}[y^2])$ and the Langlands quotient of the representation $(\varpi, z\mathbf{C}[z^2, z^{-2}])$ of $\mathfrak{sl}_2$.

Corollary 3.2. *As representations of $\mathfrak{sl}_2$, $\mathcal{L}_1$ gives the equivalence*

$$(\pi, y\mathbf{C}[y^2]) \cong (\varpi, z\mathbf{C}[z^2, z^{-2}]/z^{-1}\mathbf{C}[z^{-2}]).$$

Using Proposition 3.1, we have the following corrspondence between the eigenvalue problem and existence problem of holomorphic solutions of Heun's ordinary differential equation (see e.g. [23,24]) in some complex domains.

Theorem 3.3. *There exists a linear bijection*

$$\{u \in L^2(\mathbf{R}, \mathbf{C}^2) \mid Qu = \lambda u,\ u(-x) = u(x)\}$$
$$\xrightarrow{\sim} \{f \in \mathcal{O}(\Omega) \mid H_\lambda^+ f = 0\},$$

where Ω is a simply-connected domain in $\mathbf{C}$ such that $0, 1 \in \Omega$ while $\alpha\beta \notin \Omega$, $\mathcal{O}(\Omega)$ denotes the set of holomorphic functions on Ω, and $H_\lambda^+ = H_\lambda^+(w, \partial_w)$ is the Heun ordinary differential operator given by

$$H_\lambda^+(w, \partial_w)$$
$$:= \frac{d^2}{dw^2} + \left(\frac{\frac{1}{2}-n}{w} + \frac{-\frac{1}{2}-n}{w-1} + \frac{n+1}{w-\alpha\beta}\right)\frac{d}{dw}$$
$$+ \frac{-\frac{1}{2}(n+\frac{1}{2})w - q}{w(w-1)(w-\alpha\beta)}.$$

Here the numbers n and ν are defined through the following relation:

$$n = \frac{2\nu-3}{4}, \quad \lambda = 2\nu\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}. \tag{3}$$

The accessory parameter q can be expressed explicitly by the parameters α, β and the eigenvalue λ.

We note that the Heun operator $H_\lambda^+(w, \partial_w)$ has four regular singular points, $w = 0, 1, \alpha\beta$ and ∞. The Riemann scheme of the operator $H_\lambda^+(w, \partial_w)$ is given by

$$\begin{pmatrix} 0 & 1 & \alpha\beta & \infty & ; w \\ 0 & 0 & 0 & \frac{1}{2} & \\ n+\frac{1}{2} & n+\frac{3}{2} & -n & -(n+\frac{1}{2}) & \end{pmatrix}.$$

Here each element of the first row indicates a regular singular point of H_λ^+ and those in the second and third rows are expressing the corresponding exponents. Since it is irrelevant to the present discussion, we omit the accessory parameter q in the picture.

Let us now give another proof of Theorem 1.1. We first note that, by the linear bijection in Theorem 3.3, there is a one-to-one correspondence between the finite-type even eigenfunction of Q and a polynomial solution of the Heun differential equation $H_\lambda^+ f = 0$ (see [26]; the proof is done by a similar way to [12]). Let us assume that the multiplicity of the lowest eigenvalue E is equal to 2. This implies that there is a polynomial solution of the Heun differential equation $H_E^+ f = 0$. Then, by the Riemann scheme above one concludes that $n + \frac{1}{2} = \frac{2\nu-1}{4}$ is a nonnegative integer (see [23]). This implies $\nu \in 2\mathbf{Z}_{\geq 0} + \frac{1}{2}$. It follows from the relation (3) that $E = 2\nu\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta} \geq \frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}$. Hence, by the same reasoning based on the upper bound (2), the contradiction appears. This proves again Theorem 1.1.

Remark 3.4. The odd parity counterpart to the linear bijection in Theorem 3.3 is obtained by the Laplace transform $\mathcal{L}_a$ with $a = 2$ [12]. The Riemann scheme of the corresponding Heun's operator for the odd case is described in [13] as

$$\begin{pmatrix} 0 & 1 & \alpha\beta & \infty & ; w \\ 0 & 0 & 0 & \frac{3}{2} & \\ n & n+1 & -n-\frac{1}{2} & -n & \end{pmatrix}.$$

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Note added in proof. In a recent preprint entitled "Spectral analysis of non-commutative harmonic oscillators: The lowest eigenvalue and no crossing (2013)" (arXiv:1304.5578v1), F. Hiroshima and I. Sasaki have shown that the assumption in Theorem 1.1 actually holds, whence the simplicity of the lowest eigenvalue follows whenever $\alpha \neq \beta$.

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