Linking numbers for handlebody-links

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Abstract: As a generalization of the linking number, we construct a set of invariant numbers for two-component handlebody-links. These numbers are elementary divisors associated with the natural homomorphism from the first homology group of a component to that of the complement of another component.

Key words: Handlebody-knot; handlebody-link; linking number; elementary divisor.

1. Introduction. *A handlebody-link* is an embedding of handlebodies into a 3-manifold [1]. Especially an embedding of one handlebody into a 3-manifold is called *a handlebody-knot*. In case the genus of each component of a handlebody-link is one, it can be regarded as an ordinary link.

Handlebody-links are also regarded as *neighborhood equivalence* class of spatial graphs [2]. Two spatial graphs are neighborhood equivalent if they have isotopic regular neighborhoods. Hence handlebody-links are represented by spatial graphs whose regular neighborhoods are isotopic to the handlebody-links. In the present paper, we use this representation for handlebody-links. *A contraction move* of spatial graphs is a local transformation of spatial graphs shown in Fig. 1; contracting an edge e and its inverse move, where this move is done in an embedded disc in a 3-manifold. In [1], it is shown that two spatial graphs are neighborhood equivalent (i.e. they represent the same handlebody-links) if and only if they are transformed to each other by a sequence of contraction moves.

In the ordinary knot theory, the linking number is an elementary and important invariant for oriented two-component links. In this paper, we construct an invariant for two-component handlebody-links in a 3-sphere S^3 like the linking number for ordinary links. We consider two first homology groups for each component of the two-component handlebody-links. The invariants for two-component handlebody-links are elementary divisors of a matrix whose entries are linking numbers of closed circles corresponding to the basis of one homology group and that of the other homology group. When a genus of each component of handlebody-links is one, thus they are ordinary links, our invariants coincide with the linking number of them.

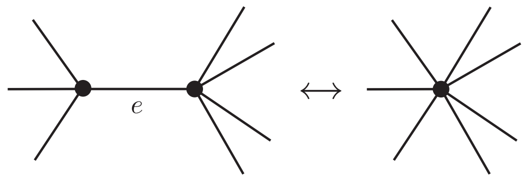

Fig. 1. A contraction move on the edge e.

The rest of the present paper is organized as follows. In Section 2, we define the invariant of two-component handlebody-links. We show an example of the invariant in Section 3. In Section 4, we see a geometrical meaning of the invariant that is a relationship to first homology groups of exterior spaces of handlebody-link components.

2. Definition. In this section, we construct the invariant numbers for two-component handlebody-links in S^3. In case both components are genus 1 (i.e. two-component links), these numbers correspond to the linking number.

Definition 2.1 (*Linking numbers for handlebody-links*)**.** Let L be a two-component handlebody-link in S^3, h_1 and h_2 be its components and m, n be genera of them respectively. We fix bases of first homology groups $H_1(h_1)$ and $H_1(h_2)$ of h_1 and h_2; $e_1, \ldots, e_m$ for $H_1(h_1)$ and $f_1, \ldots, f_n$ for $H_1(h_2)$. e_i and f_j can be regarded as embedded closed oriented circles in the S^3. Then we make a $m \times n$ matrix M whose (i, j) entry is a linking number of e_i and f_j:

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$$\begin{pmatrix} lk(e_1, f_1) & \cdots & lk(e_1, f_n) \\ \vdots & \ddots & \vdots \\ lk(e_m, f_1) & \cdots & lk(e_m, f_n) \end{pmatrix}.$$

Then we consider the elementary divisors $d_1|d_2|\dots|d_l$ (for some $0 \leq l \leq \min(m, n)$, $d_i \in \mathbf{Z}$) of M as $\mathbf{Z}$-module. Note that $d_1, d_2, \dots, d_l$ are unique up to signs. If $0 < l$, choosing positive signs, we define

$$Lk(h_1, h_2) = \{|d_1|, |d_2|, \dots, |d_l|\},$$

otherwise $Lk(h_1, h_2) = \{0\}$.

If h_1 and h_2 are separated, the linking numbers $Lk(h_1, h_2)$ is equal to $\{0\}$. We remark that the linking number above $lk(\cdot, \cdot) : H_1(h_1) \times H_1(h_2) \to \mathbf{Z}$ is bilinear for sums of elements of the first homology groups and scalar multiplying over $\mathbf{Z}$, i.e. $lk(cx_1 + dx_2, y) = c\,lk(x_1, y) + d\,lk(x_2, y)$, for $x_1, x_2 \in H_1(h_1), y \in H_1(h_2)$ and $c, d \in \mathbf{Z}$, and the same relation holds for the second argument.

Theorem 2.2. *$Lk(h_1, h_2)$ is independent of the ways taking bases of $H_1(h_1)$ and $H_1(h_2)$. Thus $Lk(h_1, h_2)$ is an invariant of two-component handlebody-links.*

Proof. Changing a basis of the first homology group $H_1(h_1)$ (resp. $H_1(h_2)$) causes multiplying an $m \times m$ (resp. an $n \times n$) regular matrix with integer entries from left (resp. right) to M, and these operations do not change the elementary divisors of M. □

3. Example. We show a calculation of $Lk(h_1, h_2)$ for the handlebody-link $L = h_1 \cup h_2$ in Fig. 2, where we represent components of the handlebody-link by bouquet graphs. The original handlebody-link is a regular neighborhood of the graphs. By virtue of the bouquet graph representation, we see bases of first homology groups $H_1(h_1)$ and $H_1(h_2)$ of the components explicitly as the oriented loop edges e_1, e_2, e_3 and f_1, f_2, f_3, f_4 of the graphs.

Example 3.1. A matrix M of linking numbers of the bases of the first homology groups of h_1 and h_2 is calculated as

$$\begin{pmatrix} -1 & -1 & 0 & 2 \\ 1 & -3 & -2 & 0 \\ 0 & 0 & 2 & -2 \end{pmatrix}.$$

The elementary divisors of M are $1 \mid 2 \mid 4$ up to signs. Thus $Lk(h_1, h_2) = \{1, 2, 4\}$.

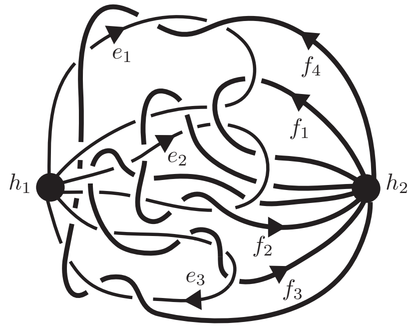

Fig. 2. A handlebody-link $L = h_1 \cup h_2$.

We remark that by sliding a loop along another loop, we have a new bouquet graph representing the same handlebody-link. This corresponds to adding or subtracting a row or column to other row or column. Therefore we can choose bouquet graphs such that the non-zero entries of the linking matrix of their loops is exactly the elementary divisors of the matrix. Thus the linking number of the handlebody-link is realized by such bouquet graphs.

4. Geometrical meanings of $\boldsymbol{Lk(h_1, h_2)}$. In this section, we show a relationship between the linking numbers $Lk(h_1, h_2)$ of handlebody-links and the first homology group of a complement space of one component h_1 (resp. h_2) in S^3 with relations derived from another component h_2 (resp. h_1).

First we consider the first homology group $H_1(S^3 \setminus h_1)$ of the complement of h_1. Remark that $H_1(S^3 \setminus h_1) = \mathbf{Z}^m$ where m is a genus of h_1. Let $\{f_1, f_2, \dots, f_n\}$ be a basis of the first homology group $H_1(h_2)$ of h_2 where n is a genus of h_2. Since h_2 is in the complement space $S^3 \setminus h_1$, we can regard the elements of the basis as oriented closed circles in $S^3 \setminus h_1$ and as elements of $H_1(S^3 \setminus h_1)$. Then we consider a quotient group of $H_1(S^3 \setminus h_1)$ by a subgroup generated by $f_1, f_2, \dots, f_n$;

$$A_1 = H_1(S^3 \setminus h_1)/\langle f_1, f_2, \dots, f_n \rangle.$$

A_1 is independent of choices of the basis of $H_1(h_2)$ and determined only by L. A_1 is a finitely generated abelian group and have a form

$$A_1 = \mathbf{Z}^{m-k} \oplus \mathrm{Tor}(A_1),$$

for some k, where $\mathrm{Tor}(A_1)$ is a torsion part of A_1. If the component h_2 is separated from h_1, the circles of the basis of $H_1(h_2)$ are trivial circles in $S^3 \setminus h_1$ and $A_1 = H_1(S^3 \setminus h_1) = \mathbf{Z}^m$. By tangling h_2 to h_1, a part of $\mathbf{Z}^m$ degenerates to the torsion part $\mathrm{Tor}(A_1)$. So we think $\mathrm{Tor}(A_1)$ has information of a tangle between h_1 and h_2.

On the other hand, we can construct a group A_2 as above exchanging the roles of h_1 and h_2,

$$A_2 = \mathbf{Z}^{n-k'} \oplus \mathrm{Tor}(A_2),$$

for some k', where $\mathrm{Tor}(A_2)$ is a torsion part of A_2. $\mathrm{Tor}(A_2)$ also has information of a tangle between h_1 and h_2. Then we have the next proposition.

Proposition 4.1. *Suppose that m is not smaller than n. Then $A_1 \cong \mathbf{Z}^{m-n} \oplus A_2$.*

Proof. It is sufficient to show $\mathrm{Tor}(A_1) \cong \mathrm{Tor}(A_2)$. We see a form of $\mathrm{Tor}(A_1)$. Let Γ_1 and Γ_2 be bouquet graphs representing the components h_1 and h_2 respectively, i.e. the regular neighborhood of Γ_1 (resp. Γ_2) is equivalent to h_1 (resp. h_2). Let $\{e_1, e_2, \ldots, e_m\}$ and $\{f_1, f_2, \ldots, f_n\}$ be edges of Γ_1 and Γ_2. They are regarded as bases of $H_1(h_1)$ and $H_1(h_2)$. We take a basis $\{x_1, x_2, \ldots, x_m\}$ of $H_1(S^3 \setminus h_1) = H_1(S^3 \setminus \Gamma_1) = \mathbf{Z}^m$ such that x_i is a closed oriented circle that rounds the edge e_i of Γ_1 one time. Each f_j, as a close oriented circle in $S^3 \setminus h_1$, is represented by the basis of $H_1(S^3 \setminus \Gamma_1)$ as

$$f_j = a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m, \quad a_{ij} \in \mathbf{Z}, \quad (1 \le j \le n).$$

From the choice of basis, the coefficient a_{ij} is how many times the edge f_j of Γ_2 rounds the edge e_i of Γ_1. These observations lead a representation of $A_1 = H_1(S^3 \setminus h_1)/\langle f_1, f_2, \ldots, f_n \rangle$;

$$\mathbf{Z}^n \xrightarrow{\psi} \mathbf{Z}^m (= H_1(S^3 \setminus h_1)) \xrightarrow{\varphi} A_1 \longrightarrow 0,$$

which is an exact sequence of $\mathbf{Z}$-module homomorphisms such that $\mathrm{Im}\,\psi = \ker\varphi$ and φ is a surjection, where a matrix representation for ψ is equal to M in Section 2 (assuming appropriate orientations for edges of Γ_1 and Γ_2). By appropriate change of basis of $H_1(S^3 \setminus h_1)$ and relations for them (i.e. multiplying regular matrices from both sides of M), we have a rectangular diagonal matrix

$$\begin{pmatrix} d_1 & & & & & \\ & d_2 & & & O & \\ & & \ddots & & & \\ & & & d_l & & \\ & & & & 0 & \\ & O & & & & \ddots \end{pmatrix},$$

where $d_1 | d_2 | \cdots | d_l$ are elementary divisors of M. Therefore A_1 have a form

$$A_1 = \mathbf{Z}^{m-l} \oplus \mathbf{Z}/d_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/d_l\mathbf{Z}$$

and we have

$$\mathrm{Tor}(A_1) = \mathbf{Z}/d_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/d_l\mathbf{Z}.$$

We do the same operation for $H_1(S^3 \setminus h_2)$ as above and have a representation of the abelian group A_2;

$$\mathbf{Z}^m \xrightarrow{\psi'} \mathbf{Z}^n (= H_1(S^3 \setminus h_2)) \xrightarrow{\varphi'} A_2 \longrightarrow 0,$$

where a matrix representation for ψ' is ${}^tM$, a transposed matrix of M. Since elementary divisors of ${}^tM$ is equal to those of M, we have a form of A_2,

$$A_2 = \mathbf{Z}^{n-l} \oplus \mathbf{Z}/d_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/d_l\mathbf{Z}$$

and we have

$$\mathrm{Tor}(A_2) = \mathbf{Z}/d_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/d_l\mathbf{Z}.$$

Therefore $\mathrm{Tor}(A_1) \cong \mathrm{Tor}(A_2)$. □

A_1 and A_2 have the same torsion part $\mathbf{Z}/d_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/d_l\mathbf{Z}$, which have the information of tangle of h_1 and h_2, and this is what the linking numbers $Lk(h_1, h_2)$ indicate. We remark that if $d_i = 1$, $\mathbf{Z}/d_i\mathbf{Z}$ is a trivial group and does not appear explicitly in the torsion part. When we calculate the linking number from the torsion part, we also need the number l which is the difference of rank between A_1 and $H_1(S^3 \setminus h_1)$ or A_2 and $H_1(S^3 \setminus h_2)$.

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