The existence of solutions for Hénon equation in hyperbolic space

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Abstract: In this paper, we use the variational methods to study the following problem

$$-\Delta_{\mathbf{B}^N} u = (d(x))^\alpha |u|^{p-2}u, u \in H_r^1(\mathbf{B}^N) \tag{1}$$

in Hyperbolic space $\mathbf{B}^N$, where $\alpha > 0$, $d(x) = d_{\mathbf{B}^N}(0, x)$, and $H_r^1(\mathbf{B}^N)$ denote the Sobolev space of radial $H^1(\mathbf{B}^N)$ function on the disc model of the Hyperbolic space $\mathbf{B}^N$ and $\Delta_{\mathbf{B}^N}$ denotes the Laplace-Beltrami operator on $\mathbf{B}^N$, $N \geq 3$. Unlike the corresponding problem in Euclidean space $\mathbf{R}^N$, we prove that there exists a positive solution of problem (1) provided that $p \in (2, \frac{2N+2\alpha}{N-2})$ which will be contrasted with a result due to Gidas and Spruck [6].

Key words: Hénon equation; mountain pass theorem; hyperbolic symmetry solution; hyperbolic space.

1. Introduction and main result. In this paper, we study the existence of positive solution for the following problem

$$-\Delta_{\mathbf{B}^N} u = (d(x))^\alpha |u|^{p-2}u, u \in H_r^1(\mathbf{B}^N) \tag{2}$$

where $\alpha > 0$, $p \in (2, \frac{2N+2\alpha}{N-2})$, $d(x) = d_{\mathbf{B}^N}(0, x)$, and $H_r^1(\mathbf{B}^N)$ denote the Sobolev space of radial $H^1(\mathbf{B}^N)$ function on the disc model of the Hyperbolic space $\mathbf{B}^N$ and $\Delta_{\mathbf{B}^N}$ denotes the Laplace-Beltrami operator on $\mathbf{B}^N$, $N \geq 3$.

When posed in Euclidean space $\mathbf{R}^N$, problem (2) has two features. First it is the following problem

$$-\Delta u = |x|^\alpha |u|^{p-2}u \ in\ \mathbf{R}^N. \tag{3}$$

Attention was focused on the existence and Liouville-type theorem for solutions of problem (3). There is a host of later important contributions to the subject, among them we must mention the famous paper [6] where the Liouville-type theorem of problem (3) was obtained. They proved that the only non-negative solution of (3) is $u = 0$ when

$$2 < p < \frac{2N+2\alpha}{N-2}, \quad N \geq 3.$$

Second, problem (3) is known as the Hénon equation

$$-\Delta u = |x|^\alpha |u|^{p-2}u, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega, \tag{4}$$

where Ω is a unit ball in $\mathbf{R}^N$. Equation (4) was proposed by M. Hénon in [8] when he studied rotating stellar structures and is called Hénon equation. A standard compactness argument show that the infimum

$$\inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\, dx}{(\int_\Omega |x|^\alpha |u|^p\, dx)^{\frac{2}{p}}} \tag{5}$$

is achieved for any $2 < p < 2^*$ and $\alpha > 0$. In 1982, Ni proved in [10] that the infimum

$$\inf_{u \in H_{0,rad}^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\, dx}{(\int_\Omega |x|^\alpha |u|^p\, dx)^{\frac{2}{p}}} \tag{6}$$

is achieved for any $p \in (2, \frac{2N+2\alpha}{N-2})$ by a function in $H_{0,rad}^1(\Omega)$, the space of radial $H_0^1(\Omega)$ functions. Thus, radial solutions of (3) exist also for (Sobolev) supercritical exponents p.

A natural question is whether any minimizer of (5) must be radially symmetric in the range $2 < p < \frac{2N}{N-2}$ and $\alpha > 0$. For $\alpha > 0$, Since the function $r \mapsto r^\alpha$ is increasing, neither rearrangement arguments nor the moving plane techniques of [7] can be applied. Therefore nonradial solutions could be expected. Smets et al. also proved in [13] some symmetry-breaking results for (4). They proved

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that minimizers of (5) (the so-called ground-state solutions, or least energy solutions) cannot be radial for α large enough. As a consequence, (4) has at least two solutions when α is large (see also [14]). Further results on problem (4) can be found in [3–5,14] for residual symmetry properties and asymptotic behavior of ground states (for $p \to \frac{2N}{N-2}$, or $\alpha \to \infty$) and in [1,11,12] for existence and multiplicity of nonradial solutions for critical, supercritical and slightly subcritical growth.

It is also of interest to study problem (3) and (4) with respect to different ambient geometries in particular to see how curvature properties affect the existence and nature of solutions. A recent paper by He and Wang [9] have studied the existence or nonexistence and asymptotic behavior of ground state solution of the following elliptic equation

$$(7) \qquad -\Delta_{\mathbf{B}^N} u = d(x)^\alpha |u|^{p-2} u \ in\ \Omega, u \in H_0^1(\Omega),$$

where $\Omega \subset \mathbf{B}^N$ is geodesic ball with radial 1. They proved that the ground state solution of problem (7) is nonradial as $p \to \frac{2N}{N-2}$.

However, for problem (2), there exists some difference from Euclidean space. Firstly, we are working in $\mathbf{B}^N$ which is a noncompact manifold, so it means that $H^1(\mathbf{B}^N) \hookrightarrow L^p(\mathbf{B}^N)$ is not compact for any $2 \leq p \leq \frac{2N}{N-2}$. Secondly, the weight function $d(x)$ depends on the Riemannian distance r from a pole o, we have some difficulties in proving that

$$\int_{\mathbf{B}^N} d(x)^\alpha |u(x)|^p \, dV_{\mathbf{B}^N} < \infty, \ \forall u \in H^1(\mathbf{B}^N).$$

So the functional of problem (2) is not well defined and cannot satisfy the $(PS)_c$ condition for all $c > 0$. Below we will show that we can overcome this difficulty by restricting to the radial situation.

Our main result is as follows:

Theorem 1.1. *Problem* (2) *has at least one positive solution provided that* $p \in (2, \frac{2N+2\alpha}{N-2})$, *where* $\alpha > 0$.

The proof of this result will be given in section 3. In section 2, we give some basic facts about hyperbolic space and prove that the map $u \mapsto (d(x))^{\frac{\alpha}{p}} u$ from $H_r^1(\mathbf{B}^N)$ to $L^p(\mathbf{B}^N)$ is compact for $p \in (2, x\frac{2N}{N-2-\frac{2\alpha}{p}})(2 < p < \frac{2N+2\alpha}{N-2})$. Finally, in section 4, a generalization to the generally elliptic problem is presented.

2. Preliminaries. Hyperbolic space $\mathbf{H}^N$ is a complete simple connected Riemannian manifold which has constant sectional curvature equal to -1. There are several models for $\mathbf{H}^N$ and we will use the Poincaré ball model $\mathbf{B}^N$ in this paper.

The Poincaré ball model for the hyperbolic space is:

$$\mathbf{B}^N = \{x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^N |\ |x| < 1\}$$

endowed with Riemannian metric g given by $g_{ij} = (p(x))^2 \delta_{ij}$ where $p(x) = \frac{2}{1-|x|^2}$. We denote the hyperbolic volume by $dV_{\mathbf{B}^N}$ and is given by $dV_{\mathbf{B}^N} = (p(x))^N \, dx$. The hyperbolic gradient and the Laplace Beltrami operator are:

$$\Delta_{\mathbf{B}^N} = (p(x))^{-N} div((p(x))^{N-2} \nabla u)), \quad \nabla_{\mathbf{B}^N} u = \frac{\nabla u}{p(x)}$$

where ∇ and div denote the Euclidean gradient and divergence in $\mathbf{R}^N$, respectively.

The hyperbolic distance $d_{\mathbf{B}^N}(x, y)$ between $x, y \in \mathbf{B}^N$ in the Poincaré ball model is given by the formula:

$$d_{\mathbf{B}^N}(x, y) = Arccosh\left(1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}\right).$$

From this we immediately obtain for $x \in \mathbf{B}^N$,

$$d(x) = d_{\mathbf{B}^N}(0, x) = \log\left(\frac{1+|x|}{1-|x|}\right).$$

Let $u \in H^1(\mathbf{B}^N)$, we can not prove that

$$\begin{aligned} &\textstyle\int_{\mathbf{B}^N} d(x)^\alpha |u|^p dV_{\mathbf{B}^N} \\ &\quad = \textstyle\int_{B(0,1)} \left[\log\left(\frac{1+|x|}{1-|x|}\right)\right]^\alpha |u|^p \left(\frac{2}{1-|x|^2}\right)^N \, dx < \infty. \end{aligned}$$

It implies that for $u \in H^1(\mathbf{B}^N)$, the functional

$$(8) \qquad \begin{aligned} I(u) = &\frac{1}{2}\int_{\mathbf{B}^N} |\nabla_{\mathbf{B}^N} u|^2 dV_{\mathbf{B}^N} \\ &- \frac{1}{p}\int_{\mathbf{B}^N} d(x)^\alpha (u^+)^p dV_{\mathbf{B}^N}, \end{aligned}$$

corresponding to (2) is not well defined. We also know that the embedding $H^1(\mathbf{B}^N) \hookrightarrow L^p(\mathbf{B}^N)$ is not compact for any $2 \leq p \leq \frac{2N}{N-2}$. Thus the functional of problem (2) cannot satisfy the $(PS)_c$ condition for all $c > 0$. Below we can overcome this difficulty by restricting to the radial situation.

Let $H_r^1(\mathbf{B}^N)$ denotes the subspace

$$H_r^1(\mathbf{B}^N) = \{u \in H^1(\mathbf{B}^N) : u \ is\ radial\}.$$

Since the hyperbolic sphere with central $0 \in \mathbf{B}^N$ is also a Euclidean sphere with central $0 \in \mathbf{B}^N$, $H_r^1(\mathbf{B}^N)$ can also be seen as the subspace consisting of hyperbolic radial functions.

Lemma 2.1. *Let $u \in H_r^1(\mathbf{B}^N)$, then*

$$|u(x)| \leq \frac{1}{\sqrt{\omega_{N-1}(N-2)}}\left(\frac{1-|x|^2}{2}\right)^{\frac{N-2}{2}}\frac{1}{|x|^{\frac{N-2}{2}}}\|u\|_{H^1(\mathbf{B}^N)},$$

or

$$|u(x)| \leq \frac{1}{\sqrt{\omega_{N-1}}}\left(\frac{1-|x|^2}{2}\right)^{\frac{N-1}{2}}\frac{1}{|x|^{\frac{N}{2}}}\|u\|_{H^1(\mathbf{B}^N)},$$

where ω_{N-1} is the surface area of S^{N-1}.

Proof. Let $u \in H_r^1(\mathbf{B}^N)$, then $u(x) = u(|x|)$, we have

$$\omega_{N-1}\int_0^1 u'(s)^2\left(\tfrac{2}{1-s^2}\right)^{N-2}s^{N-1}\,ds = \int_{\mathbf{B}^N}|\nabla_{\mathbf{B}^N}u|^2\,dV_{\mathbf{B}^N} < \infty,$$

where ω_{N-1} is the surface area of S^{N-1}. Thus for $u \in H_r^1(\mathbf{B}^N)$,

$$\begin{aligned}
u(|x|) &= -\int_{|x|}^1 u'(s)\,ds \\
&\leq \left(\int_0^1 (u'(s))^2\left(\tfrac{2}{1-s^2}\right)^{N-2}s^{N-1}\,ds\right)^{\frac12} \\
&\quad\cdot\left(\int_{|x|}^1\left(\tfrac{1-s^2}{2}\right)^{N-2}s^{-(N-1)}\,ds\right)^{\frac12} \\
&\leq \omega_{N-1}^{-\frac12}\|u\|_{H^1(\mathbf{B}^N)}\left(\tfrac{1-|x|^2}{2}\right)^{\frac{N-2}{2}}\frac{1}{|x|^{\frac{N}{2}}}\left(\int_{|x|}^1 s\,ds\right)^{\frac12} \\
&\leq \omega_{N-1}^{-\frac12}\|u\|_{H^1(\mathbf{B}^N)}\left(\tfrac{1-|x|^2}{2}\right)^{\frac{N-1}{2}}\frac{1}{|x|^{\frac{N}{2}}}
\end{aligned}$$

or

$$\begin{aligned}
u(|x|) &= -\int_{|x|}^1 u'(s)\,ds \\
&\leq \omega_{N-1}^{-\frac12}\|u\|_{H^1(\mathbf{B}^N)}\left(\tfrac{1-|x|^2}{2}\right)^{\frac{N-2}{2}}\left(\int_{|x|}^1 s^{-(N-1)}\,ds\right)^{\frac12} \\
&\leq (\omega_{N-1}(N-2))^{-\frac12}\|u\|_{H^1(\mathbf{B}^N)}\left(\tfrac{1-|x|^2}{2}\right)^{\frac{N-2}{2}}\frac{1}{|x|^{\frac{N-2}{2}}}.
\end{aligned}$$

□

Lemma 2.2. *The map $u \mapsto (d(x))^m u$ from $H_r^1(\mathbf{B}^N)$ to $L^p(\mathbf{B}^N)$ is compact for $p \in (2, \tilde{m})$, where $m > 0$, and*

$$\tilde{m} = \begin{cases} \dfrac{2N}{N-2-2m}, & \textit{if } m < \dfrac{N-2}{2}, \\ \infty & \textit{otherwise.} \end{cases}$$

Proof. By Lemma 2.1, we have that

$$\begin{aligned}
&\int_{\mathbf{B}^N}(d(x))^{mp}|u(x)|^p\,dV_{\mathcal{B}^N} \\
&\quad= \omega_{N-1}\int_0^1\left(\ln\tfrac{1+r}{1-r}\right)^{mp}|u(r)|^p r^{N-1}\left(\tfrac{2}{1-r^2}\right)^N dr \\
&\quad\leq \omega_{N-1}^{1-\frac{p}{2}}\|u\|_{H^1(\mathbf{B}^N)}^p \\
&\qquad\cdot\left[\int_0^{\frac12}\left(\ln\tfrac{1+r}{1-r}\right)^{mp}\left(\tfrac{1-r^2}{2}\right)^{\left(\frac{N-2}{2}p-N\right)}r^{N-1-\frac{N-2}{2}p}\,dr\right. \\
&\qquad\left.+\int_{\frac12}^1\left(\ln\tfrac{1+r}{1-r}\right)^{mp}\left(\tfrac{1-r^2}{2}\right)^{\left(\frac{N-1}{2}p-N\right)}r^{N-1-\frac{N}{2}p}\,dr\right] \\
&\quad\leq C\|u\|_{H^1(\mathbf{B}^N)}^p.
\end{aligned}$$

Indeed, for $\ln\frac{1+r}{1-r} \leq \frac{2r}{1-r^2}$, $p > 2$, we have

$$\begin{aligned}
&\int_0^{\frac12}\left(\ln\tfrac{1+r}{1-r}\right)^{mp}\left(\tfrac{1-r^2}{2}\right)^{\left(\frac{N-2}{2}p-N\right)}r^{N-1-\frac{N-2}{2}p}\,dr \\
&\quad\leq C\int_0^{\frac12} r^{N-1-\frac{N-2}{2}p+mp}\,dr \leq C,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac12}^1\left(\ln\tfrac{1+r}{1-r}\right)^{mp}\left(\tfrac{1-r^2}{2}\right)^{\left(\frac{N-1}{2}p-N\right)}r^{N-1-\frac{N}{2}p}\,dr \\
&\quad\leq C\int_{\frac12}^1\left(\ln\tfrac{1+r}{1-r}\right)^{mp}\left(\tfrac{1-r^2}{2}\right)^{\left(\frac{N-1}{2}p-N\right)}\,dr \\
&\quad\leq C\int_{\ln 3}^{\infty}s^{mp}\left(\tfrac{2e^s}{(e^s+1)^2}\right)^{\left(\frac{N-1}{2}p-N+1\right)}\,ds \leq C.
\end{aligned}$$

This show the map is continuous, for all $p \in (2, \frac{2N}{N-2-2m})$. Now we will show it is compact.

From [2], we know that $H_r^1(\mathbf{B}^N) \hookrightarrow L^q(\mathbf{B}^N)$ is compact for all $q \in (2, \frac{2N}{N-2})$. Then, by the Hölder inequality for $a \in (0,1)$,

$$\begin{aligned}
&\int_{\mathbf{B}^N} d(x)^{mp}|u(x)|^p\,dV_{\mathbf{B}^N} \\
&\quad= \int_{\mathbf{B}^N} d(x)^{mp}|u|^{p-qa}|u|^{qa}\,dV_{\mathbf{B}^N} \\
&\quad\leq \left(\int_{\mathbf{B}^N}|u|^q\,dV_{\mathbf{B}^N}\right)^a \\
&\qquad\cdot\left[\int_{\mathbf{B}^N}(d(x)^{mp}|u|^{p-qa})^{\frac{1}{1-a}}\,dV_{\mathbf{B}^N}\right]^{1-a}.
\end{aligned}$$

Now, we only need to check that

$$p^* = \frac{p-qa}{1-a} < \frac{2N}{N-2-2\frac{mp}{p-qa}}, \tag{9}$$

if $m < \frac{N-2}{2}$. It is easy to check that (9) holds if and only if

$$p(N-2-2m) < 2N(1-a) + qa(N-2). \tag{10}$$

Thus for a fixed $p < \frac{2N}{N-2-2m}$, (10) may easily be achieved by choosing a sufficiently small.

Hence, we have

$$\||d(x)|^m u\|_{L^p(\mathbf{B}^N)} \leq \|u\|_{L^q(\mathbf{B}^N)}^{\frac{a}{p}}\|u\|_{H^1(\mathbf{B}^N)}^{\frac{1-a}{p}}$$

where $a > 0$ and is small. It is easy to see that this Lemma holds. □

3. Proof of Theorem 1.1. Let us denote the energy functional corresponding to (2) by

$$I(u) = \frac{1}{2}\int_{\mathbf{B}^N} |\nabla_{\mathbf{B}^N} u|^2 dV_{\mathbf{B}^N} - \frac{1}{2^*}\int_{\mathbf{B}^N} d(x)^\alpha (u^+)^p dV_{\mathbf{B}^N} \tag{11}$$

defined on $H_r^1(\mathbf{B}^N)$. By Lemma 2.2, we know that

$$\int_{\mathbf{B}^N} (d(x))^\alpha |u(x)|^p \, dV_{\mathbf{B}^N} = \int_{\mathbf{B}^N} [(d(x))^{\frac{\alpha}{p}} |u(x)|]^p \, dV_{\mathbf{B}^N} < \infty, \ u \in H_r^1(\mathbf{B}^N).$$

Then we see that (11) is well defined and it is known that critical points of the functional $I \in C^1(H_r^1(\mathbf{B}^N), \mathbf{R})$ correspond to solutions of problem (2).

Now, we will prove Theorem 1.1.

Proof of Theorem 1.1. Define $T : H_r^1(\mathbf{B}^N) \to H_r^1(\mathbf{B}^N)$ by

$$\langle Tu, v\rangle_{H_r^1(\mathbf{B}^N)} = \int_{\mathbf{B}^N} (d(x))^\alpha |u|^{p-1} v \, dV_{\mathbf{H}^N},$$

then, $\langle -\Delta_{\mathbf{B}^N} Tu, v\rangle_{L^2} = \langle (d(x))^\alpha |u|^{p-1}, v\rangle_{L^2}, Tu = -\Delta_{\mathbf{B}^N}^{-1}((d(x))^\alpha |u|^{p-1})$. Thus, T may be decomposed as follows:

$$\begin{aligned} T : H_r^1 &\to L^{\frac{2N(p-1)}{N+2}} \to L^{\frac{2N}{N+2}} \to (H_r^1)^{-1} \to H_r^1 \\ u &\mapsto (d(x))^{\frac{\alpha}{p-1}}|u| \mapsto ((d(x))^{\frac{\alpha}{p-1}}|u|)^{p-1} \\ &\mapsto (d(x))^\alpha |u|^{p-1} \mapsto Tu. \end{aligned}$$

By Lemma 2.2 and $\frac{2N(p-1)}{N+2} < \frac{2N}{N-2-\frac{2\alpha}{p-1}}$ if $\frac{\alpha}{p-1} < \frac{N-2}{2}$ (from the hypothesis $p < \frac{2N+2\alpha}{N-2}$), then T is compact from $H_r^1(\mathbf{B}^N)$ to $H_r^1(\mathbf{B}^N)$, then $I : H_r^1(\mathbf{B}^N) \longrightarrow \mathbf{R}$ satisfies the Palais-Smale condition as [10] and using the Mountain Pass theorem similarly as [10], we can get a radial solution of problem (2). Multiplying the equation by u^- and integrating over $\mathbf{B}^N$, we find $u^- = 0$, and u is a solution of the equation (2). □

4. Further result. The method used in the proof of Theorem 1.1 can be applied to study the following problem

$$\begin{cases} -\Delta_{\mathbf{B}^N} u = K(d(x)) f(u), \\ u \in H_r^1(\mathbf{B}^N), \end{cases} \tag{12}$$

where $r = d(x)$, $K(r)$ and $f(u)$ satisfy the following hypothesis:
(i) $K(r)$ is a nonnegative continuous function with $K(0) = 0$ and $K \not\equiv 0$ in $\mathbf{B}^N$.
(ii) $K(r) = O(r^\alpha)$ at $r = 0$ and $K(r) = O(r^\alpha)$ as $r \to \infty$ for some $\alpha > 0$.
(iii) f is a continuous function, $f(u) \geq 0$ for all $u > 0$, $f(u) = o(u)$ at $u = 0$ and $\frac{f(u)}{u} \to \infty$ as $u \to \infty$.
(iv) $|f(u)| \leq C(1 + |u|)^{p-1}$, where $p < \frac{2N+2\alpha}{N-2}$ for u large.
(v) There exists constants $\theta \in (0, \frac{1}{2})$such that $F(u) = \int_0^u f(t) \, dt \leq \theta u f(u)$ for $u \in \mathbf{R}$.

Replace $(u^+)^p$ with $\tilde{F}$ where $\tilde{F}(u) = \int_0^u \tilde{f}(t) \, dt$ and $\tilde{f}(u) = 0$ for all $u \leq 0$ and $\tilde{f}(u) = f(u)$ for all $u > 0$. Then similarly as the proof of Theorem 1.1, we can also get the following result.

Theorem 4.1. *Under the hypotheses* (i)–(v)*, problem* (12) *possesses a positive solution.*

Proof. By (iii) and (iv), we have that $I(u) = \frac{1}{2}\|u\|^2 + o(\|u\|^2)$ as $u \to 0$. Then there exists $r > 0$ such that

$$b := \inf_{\|u\|=r} I(u) > 0.$$

By (v), we have that for any $u \in H_r^1(\mathbf{B}^N) \setminus \{0\}$,

$$I(tu) \leq \frac{1}{t^2}\|u\|^2 - t^{\frac{1}{\theta}} C \int_{\mathbf{B}^N} K(d(x))^\alpha |u|^{\frac{1}{\theta}} \, dV_{\mathbf{B}^N} \to -\infty$$

as $t \to \infty$. Then there exists $e = tu$ such that $\|e\| = r$ and $I(e) \leq 0$.

Now, we want to prove that I satisfies the Palais-Smale condition. Let $\{u_n\}$ be a sequence with $\{I(u_n)\} \leq M$ and $I'(u_n) \to 0$, then

$$\begin{aligned} M + \theta\|u_n\| &\geq I(u_n) - \theta\langle I'(u_n)u_n\rangle \\ &= (\tfrac{1}{2} - \theta)\|u_n\|^2 \\ &\quad + \int_{\mathbf{B}^N} K(d(x))^\alpha [\theta \tilde{f}(u_n)u_n - \tilde{F}(u_n)] \, dV_{\mathbf{B}^N} \\ &\geq (\tfrac{1}{2} - \theta)\|u_n\|^2. \end{aligned}$$

It follows that $\|u_n\|$ is bounded.

From (i) and (ii), we have that $K(r) \leq Cr^\alpha$ for all $0 < r < \infty$ for some constant $C > 0$, and by Lemma 2.2, we have that the map: $u \mapsto (K(r))^{\frac{\alpha}{p}} u$ from $H_r^1(\mathbf{B}^N)$ to $L^p(\mathbf{B}^N)$ is compact if $p \in (2, \frac{2N}{N-2-\frac{2\alpha}{p}})(2 < p < \frac{2N+2\alpha}{N-2})$. Then we can also prove that $J'(u)$ is compact where $J(u) = \int_{\mathbf{B}^N} (K(d(x)))^\alpha \tilde{F}(u) \, dV_{\mathbf{B}^N}$. This, together with $\{u_n\}$ being bounded and $I'(u_n) = u_n - J'(u_n) \to 0$ as $n \to 0$ implies that $\{u_n\}$ has a convergent subsequence. Thus, we know that I satisfies the Palais-Smale condition. Using the Mountain pass theorem, we obtain that there exists a positive solution of problem (12). □

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References

[1] M. Badiale and E. Serra, Multiplicity results for the supercritical Hénon equation, Adv. Nonlinear Stud. **4** (2004), no. 4, 453–467.

[2] M. Bhakta and K. Sandeep, Poincaré-Sobolev equations in the hyperbolic space, Calc. Var. Partial Differential Equations **44** (2012), no. 1–2, 247–269.

[3] J. Byeon and Z.-Q. Wang, On the Hénon equation: asymptotic profile of ground states. I, Ann. Inst. H. Poincaré Anal. Non Linéaire **23** (2006), no. 6, 803–828.

[4] J. Byeon and Z.-Q. Wang, On the Hénon equation: asymptotic profile of ground states. II, J. Differential Equations **216** (2005), no. 1, 78–108.

[5] D. Cao and S. Peng, The asymptotic behaviour of the ground state solutions for Hénon equation, J. Math. Anal. Appl. **278** (2003), no. 1, 1–17.

[6] B. Gidas and J. Spruck, A priori bounds for positive solutions of nonlinear elliptic equations, Comm. Partial Differential Equations **6** (1981), no. 8, 883–901.

[7] B. Gidas, W. M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbf{R}^N$, in *Mathematical analysis and applications, part A*, Adv. in Math. Suppl. Stud., 7a, Academic Press, New York, 1981, pp. 369–402.

[8] M. Hénon, Numerical experiments on the stability of spherical stellar systems, Astronom. Astrophys. Lib. **24** (1973), 229–238.

[9] H. He and W. Wang, Existence and asymptotic behavior of ground state solution for Hénon equation in hyperbolic space. (Preprint).

[10] W. M. Ni, A nonlinear Dirichlet problem on the unit ball and its applications, Indiana Univ. Math. J. **31** (1982), no. 6, 801–807.

[11] A. Pistoia and E. Serra, Multi-peak solutions for the Hénon equation with slightly subcritical growth, Math. Z. **256** (2007), no. 1, 75–97.

[12] E. Serra, Non radial positive solutions for the Hénon equation with critical growth, Calc. Var. Partial Differential Equations **23** (2005), no. 3, 301–326.

[13] D. Smets, M. Willem and J. Su, Non-radial ground states for the Hénon equation, Commun. Contemp. Math. **4** (2002), no. 3, 467–480.

[14] D. Smets and M. Willem, Partial symmetry and asymptotic behavior for some elliptic variational problems, Calc. Var. Partial Differential Equations **18** (2003), no. 1, 57–75.