Affine translation surfaces in Euclidean 3-space

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Abstract: In this paper we define affine translation surface and classify minimal affine translation surfaces in three dimensional Euclidean space.

Key words: Translation surface; mean curvature; minimal surface.

1. Introduction. In the classical theories of minimal surfaces in three dimensional Euclidean space $\mathbf{E}^3$ (simply, Euclidean 3-space $\mathbf{E}^3$), it is well known that the Scherk surface

$$z(x,y) = \frac{1}{c}\log\frac{\cos cx}{\cos cy} \tag{1.1}$$

is the minimal surface of the type

$$z(x,y) = f(x) + g(y) \tag{1.2}$$

in $\mathbf{E}^3$. Here using the standard coordinate system of Euclidean 3-space $\mathbf{E}^3$, a surface $r(u,v)$ in $\mathbf{E}^3$ will be written as $r(u,v) = (x(u,v), y(u,v), z(u,v))$. The surface which can be written as (1.2) is usually called translation surface in Euclidean 3-space $\mathbf{E}^3$ ([5], [6]).

In this note we define affine translation surfaces $z(x,y) = f(x) + g(y+ax)$ in Euclidean 3-space $\mathbf{E}^3$ and get the following main result.

Theorem A (minimal affine translation surfaces). *Let $r(x,y) = (x,y,z(x,y))$ be a minimal affine translation surface. Then, either $z(x,y)$ is linear, or can be written as*

$$z(x,y) = \frac{1}{c}\log\frac{\cos(c\sqrt{1+a^2}x)}{\cos[c(y+ax)]}, \tag{1.3}$$

where a and c are constants and $ac \neq 0$.

2. Affine translation surfaces. Let $r(u,v)$ be a regular surface with arbitrary parameter (u,v) in Euclidean 3-space $\mathbf{E}^3$. Using the standard coordinate system of $\mathbf{E}^3$ we denote the parametric representation of the surface $r(u,v)$ by

$$r(u,v) = (x,y,z) = (x(u,v), y(u,v), z(u,v)). \tag{2.1}$$

Definition 2.1. An affine translation surface in Euclidean 3-space $\mathbf{E}^3$ is defined as a parameter surface $r(u,v)$ in $\mathbf{E}^3$ which can be written as

$$\begin{aligned} r(u,v) &= (x(u,v), y(u,v), z(u,v)) \\ &= (u, v, f(u) + g(v+au)) \\ &= (x, y, f(x) + g(y+ax)) \end{aligned} \tag{2.2}$$

for some non zero constant a and functions $f(x)$ and $g(y+ax)$ ([1–3], [4], [7]).

By a direct calculation, the first fundamental form of $r(x,y)$ can be written as

$$\begin{cases} \mathrm{I} = E\mathrm{d}x^2 + 2F\mathrm{d}x\mathrm{d}y + G\mathrm{d}y^2, \\ E = 1 + (f' + ag')^2, \\ F = g'(f' + ag'), \\ G = 1 + g'^2, \end{cases} \tag{2.3}$$

where

$$\begin{cases} f' = \dfrac{\mathrm{d}f(x)}{\mathrm{d}x}, \\ g' = \dfrac{\mathrm{d}g(v)}{\mathrm{d}v} = \dfrac{\mathrm{d}g(y+ax)}{\mathrm{d}(y+ax)}, \end{cases} \tag{2.4}$$

where $v = y + ax$. The second fundamental form of $r(x,y)$ can be written as

$$\begin{cases} \mathrm{II} = L\mathrm{d}x^2 + 2M\mathrm{d}x\mathrm{d}y + N\mathrm{d}y^2, \\ L = (f'' + a^2g'')D^{-1}, \\ M = ag''D^{-1}, \\ N = g''D^{-1}, \end{cases} \tag{2.5}$$

where

$$D^2 = EG - F^2 = 1 + (f' + ag')^2 + g'^2. \tag{2.6}$$

The Gauss curvature of $r(x,y)$ can be written as

$$K = f''g''D^{-4} = \frac{f''g''}{[1 + (f' + ag')^2 + g'^2]^2}. \tag{2.7}$$

The mean curvature of $r(x,y)$ can be written as

$$H = \frac{1}{2}[f''(1 + g'^2) + g''(1 + a^2 + f'^2)]D^{-3} \tag{2.8}$$

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$$= \frac{f''(1+g'^2) + g''(1+a^2+f'^2)}{2[1+(f'+ag')^2+g'^2]^{\frac{3}{2}}}.$$

From (2.6) we have

$$(2.9)\quad DD_y = g'g'' + ag''(f'+ag') = af'g'' + g'g'' + a^2g'g'',$$

$$(2.10)\quad DD_x = aDD_y + f'f'' + af''g' = f'f'' + af''g' + a^2f'g'' + ag'g'' + a^3g'g''.$$

3. Minimal affine translation surfaces. In this section we consider minimal affine translation surfaces in Euclidean 3-space $\mathbf{E}^3$. If the mean curvature H of the affine translation surface $r(x,y)$ in $\mathbf{E}^3$ vanishes identically, from (2.8) we have

$$(3.1)\qquad f''(1+g'^2) + g''(1+a^2+f'^2) \equiv 0.$$

Then

$$(3.2)\qquad \frac{f''}{1+a^2+f'^2} + \frac{g''}{1+g'^2} = 0.$$

Differentiating (3.2) with respect to y we get

$$(3.3)\qquad \frac{\mathrm{d}}{\mathrm{d}(y+ax)}\left(\frac{g''}{1+g'^2}\right) = 0.$$

Differentiating (3.2) with respect to x we get

$$(3.4)\qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f''}{1+a^2+f'^2}\right) + a\frac{\mathrm{d}}{\mathrm{d}(y+ax)}\left(\frac{g''}{1+g'^2}\right) = 0.$$

Therefore we have

$$(3.5)\qquad \frac{f''}{1+a^2+f'^2} = -\frac{g''}{1+g'^2} = -c,$$

where c is a constant. The constant $c = 0$ means that $f'' = g'' \equiv 0$ and the affine translation surface $r(u,v)$ is a plane. Let $c \neq 0$ and solving (3.5) we get

$$(3.6)\qquad \begin{cases} f(x) = \dfrac{1}{c}\log\cos(c\sqrt{1+a^2}x), \\ g(y+ax) = -\dfrac{1}{c}\log\cos[c(y+ax)]. \end{cases}$$

Theorem 3.1. *Let $r(x,y) = (x,y,z(x,y))$ be a minimal affine translation surface. Then either $z(x,y)$ is linear or can be written as*

$$(3.7)\qquad z(x,y) = \frac{1}{c}\log\frac{\cos(c\sqrt{1+a^2}x)}{\cos[c(y+ax)]}.$$

The metric of the affine translation surface given by (3.7) is

$$\begin{aligned}(3.8)\qquad \mathrm{I} &= E\mathrm{d}x^2 + 2F\mathrm{d}x\mathrm{d}y + G\mathrm{d}y^2 \\ &= \{1 + [-\sqrt{1+a^2}\tan(c\sqrt{1+a^2}x) \\ &\quad + a\tan[c(y+ax)]]^2\}\mathrm{d}x^2 \\ &\quad + 2\tan[c(y+ax)] \\ &\quad \times [-\sqrt{1+a^2}\tan[c\sqrt{1+a^2}x] \\ &\quad + a\tan[c(y+ax)]]\mathrm{d}x\mathrm{d}y \\ &\quad + \sec^2[c(y+ax)]\mathrm{d}y^2.\end{aligned}$$

The Gauss curvature of the affine translation surface given by (3.7) is

$$(3.9)\qquad K = \frac{-c^2A^2\sec^2(cAx)\sec^2[cY]}{\sec^2[cY] + [-A\tan(cAx) + a\tan[cY]]^2}.$$

Where

$$A = \sqrt{1+a^2}$$
$$Y = y + ax.$$

Definition 3.1. The minimal affine translation surface (3.7) is called generalized Scherk surface or affine Scherk surface in Euclidean 3-space.

Remark 3.1. If $a = 0$, the minimal affine translation surface or generalized Scherk surface $r(x,y)$ given by (3.7) is the classical Scherk surface

$$(3.10)\qquad z(x,y) = \frac{1}{c}\log\frac{\cos cx}{\sin cy}.$$

In this case, the surface is translated along two orthonormal directions. Therefore the classical Scherk surface may be called minimal orthonormal translation surface in Euclidean 3-space $\mathbf{E}^3$. It is easy to see that the metrics of (3.7) and (3.10) are different (they are homothetic).

Remark 3.2. The Gauss curvature K of the generalized Scherk surface (3.7) can be written as

$$K = \frac{-c^2(1+a^2)}{A(x,y)},$$

where

$$\begin{aligned}A(x,y) &= \cos^2(c\sqrt{1+a^2}x) \\ &\quad + [-\sqrt{1+a^2}\sin(c\sqrt{1+a^2}x)\cos[c(y+ax)] \\ &\quad + a\sin[c(y+ax)]\cos(c\sqrt{1+a^2}x)]^2.\end{aligned}$$

Therefore, when

$$(3.11)\qquad \begin{cases} x = \dfrac{1}{c\sqrt{1+a^2}}\left(n\pi \pm \dfrac{\pi}{2}\right), \\ y = \dfrac{1}{c}\left(m\pi \pm \dfrac{\pi}{2}\right) - ax, \end{cases}$$

where

$$m, n = \pm 1, \pm 2, \ldots,$$

the Gauss curvature of the generalized Scherk surface tends to the infinity.

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