Spectral function of Krein's and Kotani's string in the class Γ

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Abstract: The asymptotic behavior of the spectral function of a one-dimensional second-order differential operator is discussed. We give a necessary and sufficient condition in order that the spectral function varies regularly with index 1. The condition is closely related to the class Γ which appears in the de Haan theory.

Key words: Strum-Liouville operator; spectral measure; diffusion; Krein's correspondence; de Haan theory.

1. Introduction. The aim of the present article is to improve one of the results in [2], where we discussed the asymptotic behavior of the spectral function of a generalized second-order differential operator.

By a *string* we mean a function

$$m : (-\infty, +\infty) \longrightarrow [0, +\infty]$$

which is nondecreasing, right-continuous and satisfies $m(-\infty + 0) = 0$. The Lebesgue-Stieltjes measure $dm(x)$ describes the mass-distribution of the string. For a string m, we are interested in the spectral theory of the generalized Strum-Liouville operator

$$(1) \qquad \mathcal{L} = \frac{d}{dm(x)} \frac{d}{dx}, \quad -\infty < x < \ell,$$

where

$$\ell(= \ell(m)) = \sup\{x | m(x) < \infty\} \quad (\leq +\infty).$$

Note that the operator

$$(2) \qquad \mathcal{L} = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} \quad (a(x) > 0)$$

can be rewritten in the form (1) with a suitable change of the variable under mild conditions on $a(x)$ and $b(x)$. For example,

$$\mathcal{L} = \frac{1}{2} \left(\frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} \right) = \frac{d}{2x\,dx} \left(x \frac{d}{dx} \right), \quad x > 0$$

can be written in the form

$$(3) \qquad \mathcal{L} = \frac{d}{dx^2} \frac{d}{d\log x} = \frac{d}{de^{2s}} \frac{d}{ds}, \quad s \in \mathbf{R}$$

with $s = \log x$.

We say that a string m has left boundary of *limit circle type* if, for some $c \ (< \ell)$,

$$(4) \qquad \int_{-\infty}^{c} x^2 dm(x) < \infty.$$

In [2] strings satisfying (4) are referred to as *Kotani's strings*. If $m(-0) = 0$ then (4) is trivially satisfied and such strings are called *Krein's string*. From the viewpoint of applications we are mainly interested in Krein's strings, but it is crucial that we adopt the framework of Kotani's strings. In what follows we denote by $\mathcal{M}_{\text{circ}}$ the totality of Kotani's strings excluding the trivial case where m vanishes identically.

For each $m \in \mathcal{M}_{\text{circ}}$, we can define $\varphi_\lambda(x)$, $(x < \ell)$, for every $\lambda \in \mathbf{C}$, as the unique solution of the following integral equation:

$$\varphi_\lambda(x) = 1 - \lambda \int_{-\infty}^{x} (x - y)\varphi_\lambda(y)\, dm(y), \quad x < \ell.$$

Let $L_0^2((-\infty, \ell), dm)$ denote the space of all square integrable functions f such that $\text{Supp}\,(f) \subset (-\infty, \ell)$ and, for $f \in L_0^2((-\infty, \ell), dm)$, we define the generalized Fourier transform by

$$\widehat{f}(\lambda) = \int_{-\infty}^{\ell} f(x)\, \varphi_\lambda(x)\, dm(x), \quad \lambda > 0.$$

Then a nonnegative Radon measure $\sigma(d\xi)$ on $[0, \infty)$ is called a *spectral measure* if the Plancherel identity

$$\|f\|_{L_0^2((-\infty, \ell), dm)} = \|\widehat{f}\|_{L^2([0, \infty), d\sigma)}$$

holds. S. Kotani ([3]) proved a certain one-to-one correspondence between $m \in \mathcal{M}_{\mathrm{circ}}$ and the spectral measure $\sigma(d\xi)$ on $[0, \infty)$ such that

$$\int_{[0,\infty)} \frac{\sigma(d\xi)}{\xi^2 + 1} < \infty.$$

This correspondence is an extension of M. G. Krein's, which treats the case where

$$\int_{[0,\infty)} \frac{\sigma(d\xi)}{\xi + 1} < \infty.$$

We refer to [3] or [4] for details. The function $\sigma(\xi) := \int_{[0,x]} \sigma(d\xi)$ will be referred to as the *spectral function.*
    In [2] we studied conditions on m in order that

(5) $\sigma(\xi) \sim \text{const} \times \xi^\alpha \quad (\xi \to +0).$

('$f(x) \sim g(x)$' means that $f(x)/g(x) \to 1$.) The probabilistic meaning of the problem is as follows: Since the transition density of the diffusion process corresponding to (1) can be represented as

(6) $p(t, x, y) = \displaystyle\int_{-0}^{\infty} e^{-t\xi} \varphi_\xi(x) \varphi_\xi(y) \, d\sigma(\xi),$

the condition (5) is equivalent to

$$p(t, x, y) \sim \text{const} \cdot t^{-\alpha} \quad (t \to \infty)$$

by the well-known Tauberian theorem for Laplace transforms (see [1]).

    Although [2] treated the case $0 < \alpha < 2$, the case $\alpha = 1$ was somewhat exceptional and the condition we gave there (see Theorem A below) is rather complicated. Therefore, in the present article we shall concentrate on the case $\alpha = 1$ and simplify the condition given in [2]: Since $\sigma(\xi) \equiv \xi$ corresponds to the string $m(x) = e^x$, which appeared in (3), up to a translation, we can expect that $\sigma(\xi) \sim \xi$ holds if and only if $m(x) \approx e^x$ in some sense. We shall see that such strings can be characterized by the class Γ which appears in the so-called de Haan theory.

    2. Main results. In what follows we assume that m is an element of $\mathcal{M}_{\mathrm{circ}}$ such that $\ell(m) = \infty$ and $m(\infty - 0) = \infty$. Therefore, both $m(x)$ and $m^{-1}(x)$ $(:= \inf\{u; m(u) > x\})$ are finite for all large x. Of course σ denotes the spectral function of m.
    Theorem 1. *Let $C > 0$. Then,*

(7) $\sigma(\xi) \sim C\xi \quad (\xi \to +0)$

holds if and only if

(8) $\displaystyle\lim_{\lambda \to \infty} \{m^{-1}(\lambda x) - m^{-1}(\lambda)\} = C \log x \ (\forall x > 0).$

A sufficient condition for (8) *is that $m(x)$ has continuous derivative such that*

(9) $\displaystyle\lim_{x \to \infty} (\log m(x))' = \frac{1}{C}.$

Furthermore, if $m'(x)$ is eventually nondecreasing (i.e., nondecreasing on $[A, \infty), \exists A > 0$), then (9) *is also necessary for* (8).
    The condition (8) implies

$$m^{-1}(x) \sim C \log x \ (x \to \infty)$$

but the converse does not hold in general.

    We next extend Theorem 1 so that we can treat the case where, for example, $\sigma(\xi) \sim \xi \log(1/\xi)$ $(\xi \to +0)$: A function $L : (A, \infty) \to (0, \infty)$ is said to be *slowly varying* (at ∞) if

$$\lim_{x \to \infty} \frac{L(cx)}{L(x)} = 1, \quad \forall c > 0.$$

Typical examples are $L(x) = \text{const}$, $\log x$, $\log \log x$, $\exp \sqrt{\log x}$, *etc.* A function of the form $f(x) = x^\rho L(x)$ with slowly varying L is said to be *regularly varying* with index $\rho \in \mathbf{R}$. Following [1] we denote by R_ρ the totality of regularly varying functions with index ρ. Especially, R_0 is the totality of slowly varying functions. If $L \in R_0$, then $\varphi(x) := xL(x) \in R_1$ and therefore $\varphi^{-1}(x) \in R_1$. This implies that $\varphi^{-1}(x) \sim xL^*(x)$ for some $L^* \in R_0$. Such L^* is called a *de Bruijn conjugate* of L and is unique up to asymptotic equivalence (see [1, p.78]). In other words, L^* is a function such that

$$xL(x)L^*(xL(x)) \sim x \quad (x \to \infty).$$

For example, if $L(x) = C$ then $L^*(x) = 1/C$, and if $L(x) = \log x$, then $L^*(x) = 1/\log x$. The de Bruijn conjugate of L^* is L itself.
    Theorem 2. *Let $L \in R_0$ and let L^* be its de Bruijn conjugate. Then, the following conditions are equivalent:*

(10) $\sigma(\xi) \sim \dfrac{\xi}{L(1/\xi)} \quad (\xi \to +0);$

(11) $\displaystyle\lim_{\lambda \to \infty} \frac{m^{-1}(\lambda x) - m^{-1}(\lambda)}{L^*(\lambda)} = \log x, \quad x > 0.$

A sufficient condition for (11) *is that m^{-1} has continuous derivative such that*

(12) $(m^{-1}(x))' \sim \dfrac{L^*(x)}{x} \quad (x \to \infty).$

If $(m^{-1})'$ is eventually nonincreasing, then (12) is also necessary for (11).

It is an easy calculus to see that (12) is equivalent to

(13) $(\log m(x))' \sim \dfrac{1}{L^*(m(x))}$ $(x \to \infty)$.

Probabilistically the assertion of Theorem 2 can be written as follows by Karamata's Tauberian theorem and (6).

Corollary 1. *Let X be a linear diffusion corresponding to* (1) *and let $p(t, x, y)$ be its transition density with respect to $dm(x)$. Then,*

$$p(t, x, y) \sim \frac{1}{tL(t)} \quad (t \to \infty)$$

if and only if (11).

The following theorem will be useful in applications.

Theorem 3. *Let $m_1, m_2 \in \mathcal{M}_{\mathrm{circ}}$ and suppose that $m_1(x) \sim m_2(x)$ $(x \to \infty)$. Then m_2 satisfies* (11) *if so does m_1.*

We postpone the proofs of Theorems 1–3 until Section 4 and give here a few examples, which are already proved in [2] but now the proofs are simplified greatly.

Example 4. Let $m \in \mathcal{M}_{\mathrm{circ}}$ and suppose that $m(x) \sim Ax^\gamma e^{Bx}$ $(x \to \infty)$. Then we have (7) with $C = 1/B$. Indeed by Theorem 3 we may assume that $m(x) = Ax^\gamma e^{Bx}$ for all sufficiently large x and then

$$\lim_{x \to \infty} \frac{m'(x)}{m(x)} = B.$$

Therefore, our assertion follows from Theorem 1.

Example 5. If $m(x) \sim Ae^{Bx+\sqrt{x}}$ $(x \to \infty)$, then we have (7) with $C = 1/B$.

Example 6. Suppose that $m(x) \sim Ax^\gamma e^{\sqrt{Bx}}$ as $x \to \infty$, where $A, B > 0$ and $\gamma \in \mathbf{R}$. Also let $L(x) = B/(2 \log x)$ so that $L^*(x) = 1/L(x) = (2 \log x)/B$. Then we may assume that $m(x) = Ax^\gamma e^{\sqrt{Bx}}$ for sufficiently large x, and we have

$$(\log m(x))' \sim \frac{\sqrt{B}}{2\sqrt{x}}.$$

Since

$$L^*(m(x)) = \frac{2}{B} \log m(x) \sim \frac{2}{\sqrt{B}} \sqrt{x} \quad (x \to \infty),$$

we have

$$(\log m(x))' \sim \frac{1}{L^*(m(x))} \quad (x \to \infty).$$

Thus (13) is satisfied, and by Theorem 2 we conclude that

$$\sigma(\xi) \sim \frac{\xi}{L(1/\xi)} = \frac{2}{B} \xi \log \frac{1}{\xi} \quad (\xi \to +0).$$

3. Preliminaries. We prepare some results on so called *de Haan theory*.

Definition 7. (i) The *de Haan class* Π_+ is the totality of eventually finite functions $f : (0, \infty) \to [-\infty, \infty)$ for which there exists an $L \in R_0$ such that

(14) $\displaystyle \lim_{\lambda \to \infty} \frac{f(\lambda x) - f(\lambda)}{L(\lambda)} = \log x, \quad \forall x > 0.$

(ii) The class Γ is the totality of eventually positive functions $F : \mathbf{R} \to [0, \infty)$, nondecreasing and right-continuous, for which there exists a measurable function $g : \mathbf{R} \to (0, \infty)$ such that

(15) $\displaystyle \lim_{\lambda \to \infty} \frac{F(\lambda + xg(\lambda))}{F(\lambda)} = e^x, \quad \forall x \in \mathbf{R}.$

For example, $\log x \in \Pi_+$ and $e^x \in \Gamma$ (with $g(x) = 1$). These two classes Π_+ and Γ are closely related as follows:

Proposition 8. (i) *If $f \in \Pi_+$, then $f^{-1} \in \Gamma$.* (ii) *Conversely, if $F \in \Gamma$, then $F^{-1} \in \Pi_+$.*

For the proof we refer to $[1, \mathrm{Thm}.3.10.4]$. (When f (or F) is strictly increasing and continuous, then the assertion is almost clear.)

Lemma 1. *Let m be a string given at the beginning of Section 2 and let $L \in R_0$. Also let L^* be its de Bruijin conjugate. Then the following conditions are equivalent:*

(16) $\displaystyle \lim_{\lambda \to \infty} \frac{1}{\lambda L(\lambda)} m\left(\frac{x}{L(\lambda)} + q(\lambda) \right) = e^x, \quad x > 0$

for some $q(\lambda)$.

(17) $\displaystyle \lim_{\lambda \to \infty} \frac{m^{-1}(\lambda x) - m^{-1}(\lambda)}{L^*(\lambda)} = \log x, \quad x > 0$

Proof. Consider the inverse functions of the both sides of (16). Then it can be written as

$$\lim_{\lambda \to \infty} L(\lambda)\{m^{-1}(\lambda L(\lambda)x) - q(\lambda)\} = \log x, \quad x > 0,$$

for some $q(x)$. This is also equivalent to

$$\lim_{\lambda \to \infty} L(\lambda)\{m^{-1}(\lambda L(\lambda)x) - m^{-1}(\lambda L(\lambda))\} = \log x,$$

that is

$$\lim_{\lambda \to \infty} \frac{\varphi(\lambda)}{\lambda} \{m^{-1}(\varphi(\lambda)x) - m^{-1}(\varphi(\lambda))\} = \log x,$$

where $\varphi(\lambda) = \lambda L(\lambda)$. In other words

$$\lim_{\lambda \to \infty} \frac{\lambda}{\varphi^{-1}(\lambda)} \{m^{-1}(\lambda x) - m^{-1}(\lambda)\} = \log x,$$

which is the same as (17) because $\varphi^{-1}(\lambda) \sim \lambda L^*(\lambda)$. $\square$

Lemma 2. *Let $F : \mathbf{R} \to [0, \infty)$ be a nondecreasing function such that $F(\infty) = \infty$, and let $L \in R_0$. Also let $1/L^\#$ be its de Bruijin conjugate of $1/L$ (i.e., if $\psi(x) = x/L(x)$, then $\psi^{-1}(x) \sim x/L^\#(x)$). Then the following conditions are equivalent:*

$$(18) \qquad \frac{L(\lambda)}{\lambda} F\left(\frac{x}{L(\lambda)} + q(\lambda)\right) \to e^x \quad (\exists q(\lambda)),$$

$$(19) \qquad \frac{F^{-1}(\lambda x) - F^{-1}(\lambda)}{L^\#(\lambda)} \to \log x.$$

Proof. The proof is essentially the same as that of Lemma 1.

(18)

$$\Leftrightarrow L(\lambda)\left\{F^{-1}\left(\frac{\lambda}{L(\lambda)}x\right) - F^{-1}\left(\frac{\lambda}{L(\lambda)}\right)\right\} \to \log x$$

$$\Leftrightarrow \frac{\lambda}{\psi(\lambda)} \{F^{-1}(\psi(\lambda)x) - F^{-1}(\psi(\lambda))\} \to \log x$$

$$\Leftrightarrow \frac{\psi^{-1}(\lambda)}{\lambda} \{F^{-1}(\lambda x) - F^{-1}(\lambda)\} \to \log x$$

$$\Leftrightarrow (19).$$

$\square$

4. Proofs of Theorems 1–3. For the given $m \in \mathcal{M}_{\mathrm{circ}}$ let

$$M(x) = \int_{-\infty}^{x} m(u)\,du, \quad x \in \mathbf{R};$$

$$N(x) = \int_{-\infty}^{x} M(u)\,du$$
$$\left(= \frac{1}{2}\int_{-\infty}^{x}(x-u)^2\,dm(u)\right), \quad x \in \mathbf{R}.$$

Of course these functions exist under the assumption (4) when $\ell(m) = \infty$.

The proofs of Theorems 1 and 2 are based on the following result in [2].

Theorem A. *Let $L \in R_0$. Then, (10) holds if and only if*

$$(20) \qquad \lim_{\lambda \to \infty} \frac{L(\lambda)}{\lambda} N\left(\frac{x}{L(\lambda)} + q(\lambda)\right) = e^x, \quad \forall x \in \mathbf{R}$$

for some $q(\lambda)$.

Proof of Theorem 2. We first prove the Tauberian implication. Suppose that (10) holds. Then by Theorem A (10) we have (20). By the monotone density convergence theorem (see Theorem B in Appendix), (20) implies

$$(21) \qquad \lim_{\lambda \to \infty} \frac{1}{\lambda} M\left(\frac{x}{L(\lambda)} + q(\lambda)\right) = e^x, \quad \forall x \in \mathbf{R}.$$

By the same argument (21) implies

$$(22) \qquad \lim_{\lambda \to \infty} \frac{1}{\lambda L(\lambda)} m\left(\frac{x}{L(\lambda)} + q(\lambda)\right) = e^x, \quad \forall x \in \mathbf{R}.$$

Now (22) is equivalent to (11) by Lemma 1.

Next let us prove the converse. Suppose that (11) holds. Then $m^{-1} \in \Pi_+$ and therefore by Proposition 8 we see $m \in \Gamma$, which implies $M \in \Gamma$ (see [1, Cor. 3.10.7]). Repeating the same argument we also have $N \in \Gamma$ so that $N^{-1} \in \Pi_+$: i.e.,

$$(23) \qquad \lim_{\lambda \to \infty} \frac{N^{-1}(\lambda x) - N^{-1}(\lambda)}{L_0(\lambda)} = \log x, \quad x > 0$$

for some $L_0 \in R_0$. Let $L_1 := L_0^\#(\in R_0)$. That is $1/L_1$ is the de Bruijin conjugate of $1/L_0$. Note that $L_1 = L_0^\#$ implies $L_1^\# = L_0$. So, (23) is written as

$$(24) \qquad \lim_{\lambda \to \infty} \frac{N^{-1}(\lambda x) - N^{-1}(\lambda)}{L_1^\#(\lambda)} = \log x, \quad x > 0.$$

Therefore, by Lemma 2 this implies

$$(25) \qquad \lim_{\lambda \to \infty} \frac{L_1(\lambda)}{\lambda} N\left(\frac{x}{L_1(\lambda)} + q(\lambda)\right) = e^x,$$

for some $q(x)$. By Theorem A, this implies

$$(26) \qquad \sigma(\xi) \sim \frac{\xi}{L_1(1/\xi)}, \quad (\xi \to +0).$$

Now it remains to show that $L_1(x) \sim L(x)$. Recall that we have already proved the Tauberian implication. Therefore, (26) implies

$$(27) \qquad \frac{1}{L_1^*(\lambda)} \{m^{-1}(\lambda x) - m^{-1}(\lambda)\}$$
$$\longrightarrow \log x \quad (\lambda \to \infty) \quad \forall x > 0.$$

Comparing this with (11) we see that $L_1^*(x) \sim L^*(x)$ and hence $L_1(x) \sim L(x)$, which completes the proof of the Abelina implication.

Let us next see the latter half of the theorem. Although this fact might be familiar to some people, we give the proof for the convenience of the reader: If (12), then, for every fixed $x > 0$,

$$(m^{-1}(\lambda x))' \sim \frac{L^*(\lambda x)}{\lambda x} \sim \frac{L^*(\lambda)}{\lambda x} \quad (\lambda \to \infty)$$

so that

$$(28) \qquad \frac{\lambda}{L^*(\lambda)} \, (m^{-1})'(\lambda x) \to \frac{1}{x} \quad (\lambda \to \infty),$$

the convergence being locally uniform in $x > 0$ by the well-known property of regularly varying functions. Since

$$\frac{m^{-1}(\lambda x) - m^{-1}(\lambda)}{L^*(\lambda)} = \int_1^x \frac{\lambda}{L^*(\lambda)} \, (m^{-1})'(\lambda u) \, du,$$

we can deduce from (28) that

$$\lim_{\lambda \to \infty} \frac{m^{-1}(\lambda x) - m^{-1}(\lambda)}{L^*(\lambda)} = \int_1^x \frac{du}{u} = \log x.$$

Thus we have (11). To see the converse, use the monotone density convergence theorem (see Theorem B in Appendix). $\qquad \square$

Since Theorem 1 is just a special case of Theorem 2, we omit the proof.

Proof of Theorem 3. In view of Theorem 2 and Lemma 1 it is sufficient to show that m_2 satisfies (16) if so does m_1 when $m_1(x) \sim m_2(x)$. However, this is almost trivial. $\qquad \square$

5. Appendix.

Theorem B. *Let $F_\lambda(x)$ and $F(x)$ be absolutely continuous functions on an interval I with nondecreasing (or nonincreasing) derivatives $f_\lambda(x)$ and $f(x)$, respectively. If $F_\lambda(x) \to F(x)$ $(\lambda \to \infty)$ for all $x \in I$, then $f_\lambda(x) \to f(x)$ $(\lambda \to \infty)$ at all continuity points x of f.*

Proof. Since

$$f_\lambda(x) \le \frac{1}{\epsilon} \int_x^{x+\epsilon} f_\lambda(u) \, du = \frac{1}{\epsilon} \{F_\lambda(x+\epsilon) - F_\lambda(x)\},$$

we have

$$\limsup_{\lambda \to \infty} f_\lambda(x) \le \frac{1}{\epsilon} \{F(x+\epsilon) - F(x)\}$$
$$= \frac{1}{\epsilon} \int_x^{x+\epsilon} f(u) \, du,$$

which implies, for every $\epsilon > 0$,

$$\limsup_{\lambda \to \infty} f_\lambda(x) \le f(x+\epsilon).$$

Therefore,

$$\limsup_{\lambda \to \infty} f_\lambda(x) \le f(x+0).$$

Similarly we have

$$\liminf_{\lambda \to \infty} f_\lambda(x) \ge f(x-0).$$

$\qquad \square$

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