A note on linear independence of polylogarithms over the rationals

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Abstract: In this article, we give a new lower bound for the dimension of the linear space over the rationals spanned by 1 and values of polylogarithmic functions at a non-zero rational number. Our proof uses Padé approximation following the argument of T. Rivoal, however we adapt a new linear independence criterion due to S. Fischler and W. Zudilin. We also present an example of the linear space of dimension $\geqslant 3$ over $\mathbf{Q}$, which is generated by 1 and polylogarithms.

Key words: Polylogarithms; Padé approximation; irrationality; linear independence.

1. Introduction. For $s = 1, 2, \cdots$, consider the polylogarithmic function $\mathrm{Li}_s(z)$ defined by

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, z \in \mathbf{C}, |z| \leqslant 1 \quad (z \neq 1 \text{ if } s = 1).$$

The function satisfies $\mathrm{Li}_1(z) = -\log(1-z) = \int_0^z \frac{dt}{1-t}$ and $\mathrm{Li}_{s+1}(z) = \int_0^z \frac{\mathrm{Li}_s(t)}{t} dt$. We restrict ourselves to the case $z \in \mathbf{R}$, hence the values of polylogarithmic functions (so-called polylogarithms) are real numbers. Concerning known properties of the function, see for example [3]. E. M. Nikišin [6] and M. Hata [2] investigated sufficient conditions such that for a rational number α, the values of polylogarithmic functions $\mathrm{Li}_1(\alpha)$, $\mathrm{Li}_2(\alpha), \cdots, \mathrm{Li}_s(\alpha)$ and 1 are linearly independent over $\mathbf{Q}$. In 2003, T. Rivoal [7] showed a linear independence result of polylogarithms, stated as follows.

Theorem A (Rivoal). *Let s be an integer $\geqslant 2$. Let $\alpha = p/q \in \mathbf{Q}$ with $p, q \in \mathbf{Z}, \gcd(p, q) = 1$ and $0 < |\alpha| < 1$. For any $\varepsilon > 0$, there exists an integer $A(\varepsilon, p, q) \geqslant 1$ satisfying the following property. If $s \geqslant A(\varepsilon, p, q)$, we have*

$$\dim_{\mathbf{Q}}\{\mathbf{Q} + \mathbf{Q}\mathrm{Li}_1(\alpha) + \cdots + \mathbf{Q}\mathrm{Li}_s(\alpha)\} \geqslant \frac{1-\varepsilon}{1+\log(2)} \log(s).$$

A simple corollary of Theorem A is given by:

Corollary B (Rivoal). *For any $\alpha \in \mathbf{Q}$ with $0 < |\alpha| < 1$, the set $\{\mathrm{Li}_s(\alpha) : s \geqslant 1\}$ contains infinitely many irrational numbers.*

In Rivoal's works, it is remarkable that his statements are valid for any $\alpha \in \mathbf{Q}$ with $0 < |\alpha| < 1$, which differs from all the previous results. The proof of Theorem A is based on an important linear independence criterion due to Yu. V. Nesterenko [5]. In 2006, R. Marcovecchio [4] generalized Theorem A for an algebraic number α.

S. Fischler and W. Zudilin [1] gave in 2010 a refinement of Nesterenko's criterion by means of geometry of numbers, which is the next theorem.

Theorem C (Fischler-Zudilin). *Let $s \geqslant 1$ be an integer, and $\xi_0, \cdots, \xi_s$ be real numbers. Let $\tau > 0$ and $\gamma_1, \cdots \gamma_s \geqslant 0$. For $i \in \{0, \cdots, s\}$ and $n = 1, 2, \cdots$, consider an integer sequence $\ell_{i,n} \in \mathbf{Z}$. For $i \in \{1, \cdots, s\}$ and $n = 1, 2, \cdots$, let $\delta_{i,n} \in \mathbf{Z}$ be a positive divisor of $\ell_{i,n}$ satisfying both of* (i) *and* (ii)*:*

(i) *$\delta_{i,n}$ divides $\delta_{i+1,n}$ for any $n \geqslant 1$ and for any $i \in \{1, \cdots, s-1\}$,*

(ii) *$\dfrac{\delta_{j,n}}{\delta_{i,n}}$ divides $\dfrac{\delta_{j,n+1}}{\delta_{i,n+1}}$ for any $n \geqslant 1$ and for any $0 \leqslant i < j \leqslant s$ with $\delta_{0,n} = 1$.*

Assume moreover that there exists an increasing sequence $(Q_n)_{n\geqslant 1}$ of integers such that all of the following conditions are fulfilled as $n \to \infty$:

$$(1) \qquad Q_{n+1} = Q_n^{1+o(1)},$$

$$(2) \qquad \max_{0\leqslant i\leqslant s} |\ell_{i,n}| \leqslant Q_n^{1+o(1)},$$

$$(3) \qquad \left| \sum_{i=0}^{s} \ell_{i,n}\xi_i \right| = Q_n^{-\tau+o(1)},$$

$$(4) \qquad \delta_{i,n} = Q_n^{\gamma_i+o(1)} \text{ for any } i \in \{1, \cdots, s\}.$$

Let $M = \dim_{\mathbf{Q}}(\mathbf{Q}\xi_0 + \mathbf{Q}\xi_1 + \cdots + \mathbf{Q}\xi_s) - 1$. Then we have

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$$M \geqslant \tau + \gamma_1 + \cdots + \gamma_M.$$

It should be noted that the right-hand side of the conclusion of Theorem C also contains M. In [1], it is also achieved to give explicit sufficient conditions to show that at least 3 values of the Riemann zeta funciton are linearly independent.

Theorem D (Fischler-Zudilin)**.** *Let $s \geqslant 1$ be an integer, and $\xi_0, \cdots, \xi_s$ be real numbers. Consider real numbers $0 < A < 1, B > 1$. For any $n \geqslant 1$, let $\ell_{0,n}, \cdots, \ell_{s,n} \in \mathbf{Z}$ be integers such that*

$$\lim_{n\to\infty}\left|\sum_{i=0}^{s} \ell_{i,n}\xi_i\right|^{1/n} = A \tag{5}$$

and

$$\lim_{n\to\infty}\sup\left|\ell_{i,n}\right|^{1/n} \leqslant B \tag{6}$$

for any $i \in \{0, \cdots, s\}$. For any $n \geqslant 1$, let δ_n be a common positive divisor of $\ell_{1,n}, \cdots, \ell_{s,n}$. Assume moreover

$$AB < \lim_{n\to\infty}\inf(\gcd(\delta_n, \delta_{n+1}))^{1/n}. \tag{7}$$

Then

$$\dim_{\mathbf{Q}}(\mathbf{Q}\xi_0 + \mathbf{Q}\xi_1 + \cdots + \mathbf{Q}\xi_s) \geqslant 3.$$

2. Main theorems. We adopt the criterion not to the Riemann zeta function but to the polylogarithmc function. We rely on Theorem C and Theorem D instead of Nesterenko's linear independence criterion, by following Rivoal's argument of Padé approximation [7]. First we begin with a simple statement.

Theorem 1. *Le $s \geqslant 356$. Then for $\alpha = \frac{p}{q} \in \mathbf{Q}$, with $p, q \in \mathbf{Z}$, $\gcd(p,q) = 1$, $0 < |\alpha| < 1$, $1 \leqslant |p| \leqslant 49$, $2 \leqslant |q| \leqslant 50$, we have*

$$\dim_{\mathbf{Q}}\{\mathbf{Q} + \mathbf{Q}\mathrm{L}i_1(\alpha) + \cdots + \mathbf{Q}\mathrm{L}i_s(\alpha)\} \geqslant 3.$$

The new part of the statement comes from the refinement done in Theorem D and also our new choice of parameters. A more general statement is as follows.

Theorem 2. *Let s be an integer $\geqslant 2$. Let $\alpha = \frac{p}{q} \in \mathbf{Q}$ with p, $q \in \mathbf{Z}$, $\gcd(p,q) = 1$ and $0 < |\alpha| < 1$. Put*

$$M = \dim_{\mathbf{Q}}\{\mathbf{Q} + \mathbf{Q}\mathrm{L}i_1(\alpha) + \cdots + \mathbf{Q}\mathrm{L}i_s(\alpha)\} - 1.$$

Let $r \in \mathbf{Z}$, $1 \leqslant r < M$ defined by

$$r = \max\left\{1, \left[\frac{M}{(\log\max\{3, M\})^{\rho}}\right]\right\} \tag{8}$$

where $\rho > 0$ arbitrarily chosen and fixed, with $[a]$ the largest integer part $\leqslant a$ (floor function). Then we have

$$M \geqslant \frac{\log r + \frac{(M-1)}{2} - \frac{\log|p|}{M} - \frac{r}{M}\log r}{1 + \log 2 + \frac{\log|q|}{M} + \left(\frac{r+1}{M}\right)\log 2 + \frac{r}{M}\log r}.$$

We should note that the right-hand side of the conclusion of Theorem 2 contains M as in the statement of Theorem C. Indeed, when we substract $\frac{M-1}{2}$ from the numerator of the right-hand side and add this part on the left-hand side, then it gives only an asymptotic formula for M.

3. Proof of Theorem 2. Now we start the proof of Theorem 2. Let $1 \leqslant r < s$, r, $s \in \mathbf{Z}$, $1 \leqslant n \in \mathbf{Z}$. For $z \in \mathbf{C}$, $|z| > 1$, consider $N_n(z)$ as follows.

$$\begin{aligned}N_n(z) &= n!^{s-r}\sum_{k=1}^{\infty}\frac{(k-1)(k-2)\cdots(k-rn)}{k^s(k+1)^s\cdots(k+n)^s}z^{-k}\\ &= n!^{s-r}\sum_{k=1}^{\infty}\frac{(k-rn)_{rn}}{(k)_{n+1}^s}z^{-k}\end{aligned}$$

where $(a)_k$ is the Pochhammer symbol defined by

$$(a)_0 = 1, (a)_k = a(a+1)(a+2)\cdots(a+k-1)$$

$(k = 1, 2, \cdots)$.

Next we recall Lemma 1 of [7] as follows.

Lemma 3.

$$N_n(z) = \frac{(rn)!}{n!^r}\int_{[0,1]^s}\left(\frac{\prod_{\sigma=1}^{s} x_\sigma^r(1-x_\sigma)}{(z - x_1\cdots x_s)^r}\right)^n\frac{dx_1\cdots dx_s}{z - x_1\cdots x_s}.$$

Proof. This function is a generalized nearly-poised hypergeometric series. We do an iteration of Euler's identity of integral (*see* [9], page 108 (4.1.3)) for the function:

$$N_n(z) = z^{-rn-1}\frac{(rn)!}{n!^r}\frac{\Gamma(rn+1)^s\Gamma(n+1)^s}{\Gamma((r+1)n+2)^s}$$
$$\times {}_{s+1}F_s\left(\begin{matrix} rn+1, rn+1, \cdots, rn+1 \\ (r+1)n+2, \cdots, (r+1)n+2\end{matrix}\;\middle|\; z^{-1}\right)$$

to obtain the statement. □

Lemma 4. *Consider the differential operator $D_\lambda = \frac{1}{\lambda!}\frac{d^\lambda}{dt^\lambda}$. Define $R_n(t)$ by*

$$R_n(t) = n!^{s-r} \frac{(t-rn)_{rn}}{(t)_{n+1}^s}.$$

Then for $\sigma \in \{1,2,\cdots,s\}, j \in \{0,1,\cdots,n\}$, *we have*

$$R_n(t) = \sum_{\sigma=1}^{s} \sum_{j=0}^{n} \frac{c_{\sigma,j,n}}{(t+j)^\sigma}$$

where $c_{\sigma,j,n} = D_{s-\sigma}(R_n(t)(t+j)^s)|_{t=-j} \in \mathbf{Q}$.

Proof. This is done by decomposition into partial fractions. □

Lemma 5. *Consider* $c_{\sigma,j,n}$ *in Lemma 4 for* $\sigma \in \{1,2,\cdots,s\}$, $j \in \{0,1,\cdots,n\}$. *Write*

$$P_{0,n}(z) = -\sum_{\sigma=1}^{s} \sum_{j=1}^{n} c_{\sigma,j,n} \sum_{k=1}^{j} \frac{1}{k^\sigma} z^{j-k}$$

and

$$P_{\sigma,n}(z) = \sum_{j=0}^{n} c_{\sigma,j,n} z^j.$$

Then for any $z \in \mathbf{C}, |z| > 1$, *we have*

$$N_n(z) = P_{0,n}(z) + \sum_{\sigma=1}^{s} P_{\sigma,n}(z)\mathrm{Li}_\sigma(1/z). \tag{9}$$

Proof. We use Lemma 4 to rewrite $N_n(z)$. We then obtain:

$$\begin{aligned} N_n(z) &= \sum_{\sigma=1}^{s} \sum_{j=0}^{n} c_{\sigma,j,n} \sum_{k=1}^{\infty} \frac{1}{z^k} \frac{1}{(k+j)^\sigma} \\ &= \sum_{\sigma=1}^{s} \sum_{j=0}^{n} c_{\sigma,j,n} z^j \left(\sum_{k=1}^{\infty} \frac{1}{z^k} \frac{1}{k^\sigma} - \sum_{k=1}^{j} \frac{1}{z^k} \frac{1}{k^\sigma} \right) \\ &= \sum_{\sigma=1}^{s} \mathrm{Li}_\sigma(1/z) \sum_{j=0}^{n} c_{\sigma,j,n} z^j \\ &\quad - \sum_{\sigma=1}^{s} \sum_{j=1}^{n} c_{\sigma,j,n} \sum_{k=1}^{j} \frac{1}{k^\sigma} z^{j-k}. \end{aligned}$$

Hence the lemma follows. □

Next, be means of the integral representation, we show that $|N_n(z)|^{1/n}$ is small enough for sufficiently large n.

Lemma 6. *For any* $z \in \mathbf{R}, |z| > 1$, *the sequence* $|N_n(z)|^{1/n}$ *has a limit point. Write* $\varphi_{r,s}(z) = \lim_{n\to\infty} |N_n(z)|^{1/n}$. *Then we have*

$$0 < \varphi_{r,s}(z) \leqslant \frac{1}{|z|^r r^{s-r}}. \tag{10}$$

Proof. Stirling's formula implies

$$\lim_{n\to\infty} \left(\frac{(rn)!}{n!^r} \right)^{1/n} = r^r.$$

Since $z \in \mathbf{R}, |z| > 1$, Lemma 3 in [7] implies

$$\lim_{n\to\infty} |N_n(z)|^{1/n} = r^r \left| \frac{\prod_{\sigma=1}^{s} v_\sigma^r (1-v_\sigma)}{(z - v_1 v_2 \cdots v_s)^r} \right| > 0.$$

We now give an upper bound for $\varphi_{r,s}(z)$.
When $k \geqslant rn+1$, we get:

$$\begin{aligned} &R_n(k)|z|^{-k} \\ &= n!^{s-r} \frac{(k-rn)_{rn}}{(k)_{n+1}^s} |z|^{-k} \leqslant n^{(s-r)n} \frac{k^{rn}}{k^{s(n+1)}} |z|^{-rn} \\ &= \left(\frac{n}{k}\right)^{(s-r)n} |z|^{-rn} \frac{1}{k^s} \leqslant \left(\frac{1}{|z|^r r^{s-r}} \right)^n \frac{1}{k^s}. \end{aligned}$$

By noting $R_n(k) = 0$ when $k = 0, \cdots, rn$, we have

$$\begin{aligned} |N_n(z)| &\leqslant \sum_{k=rn+1}^{\infty} R_n(k)|z|^{-k} \\ &\leqslant \left(\frac{1}{|z|^r r^{s-r}} \right)^n \sum_{k=rn+1}^{\infty} \frac{1}{k^s} \end{aligned}$$

which implies the statement. □

Lemma 7. *For any* $\sigma \in \{0,1,\cdots,s\}$ *and* $z \in \mathbf{C}, |z| > 1$, *we have*

$$\limsup_{n\to\infty} \left| P_{\sigma,n}(z) \right|^{1/n} \leqslant r^r 2^{s+r+1} |z|. \tag{11}$$

Proof. Cauchy integral formula allows us:

$$c_{\sigma,j,n} = \frac{1}{2\pi i} \int_{|z+j|=1/2} R_n(z)(z+j)^{\sigma-1} dz.$$

On the circle $z \in \{z : |z+j| = 1/2\}$, we have

$$|(z-rn)_{rn}| \leqslant (j+2)_{rn}$$

and $|(z)_{n+1}| \geqslant 2^{-3}(j-1)!(n-j-1)!$.

Therefore (with a correction of Lemma 4 in [7]) we get:

$$\begin{aligned} |c_{\sigma,j,n}| &\leqslant n!^{s-r} \frac{(j+2)_{rn}}{(2^{-3}(j-1)!(n-j-1)!)^s} \cdot 1^{\sigma-1} \\ &= \binom{rn+j}{j} \binom{n}{j}^s \frac{(rn)!}{n!^r} \\ &\quad \times \frac{j^s (n-j)^s (rn+j+1)}{j+1} \times 8^s. \end{aligned}$$

Since we have

$$\binom{rn+j}{j} \leqslant 2^{rn+j}, \ \binom{n}{j} \leqslant 2^n, \ \frac{(rn)!}{n!^r} \leqslant r^{rn},$$

we then obtain

$$\begin{aligned} |c_{\sigma,j,n}| &\leqslant 2^{rn+j} 2^{ns} r^{rn} \frac{j^s (n-j)^s (rn+j+1)}{j+1} \cdot 8^s \\ &\leqslant (r^r 2^{s+r+1})^n (j(n-j)(rn+j+1))^s \end{aligned}$$

which yields

$$\limsup_{n\to\infty} |c_{\sigma,j,n}|^{1/n} \leqslant r^r 2^{s+r+1}.$$

We finally have, for $\sigma \in \{1, \cdots, s\}$,

$$|P_{\sigma,n}(z)| = \left|\sum_{j=0}^{n} c_{\sigma,j,n} z^j\right| \leqslant (n+1) \max_{0\leqslant j\leqslant n} |c_{\sigma,j,n} z^n|$$

therefore we have $\limsup_{n\to\infty} |P_{\sigma,n}(z)|^{1/n} \leqslant r^r 2^{s+r+1}|z|$.

Similarly, for $P_{0,n} = -\sum_{\sigma=1}^{s}\sum_{j=0}^{n} c_{\sigma,j,n} \sum_{k=1}^{j} \frac{z^{j-\sigma}}{k^\sigma}$, we have

$$\left|\sum_{k=1}^{j} \frac{z^{j-\sigma}}{k^\sigma}\right| \leqslant |z|^n \sum_{k=1}^{j} \frac{1}{k^\sigma} \leqslant n \cdot |z|^n.$$

Hence

$$\limsup_{n\to\infty} \left|P_{0,n}(z)\right|^{1/n} \leqslant r^r 2^{s+r+1}|z|.$$

The statement is achieved. □

3.1. Divisors. Put $d_n = \mathrm{lcm}(1, 2, \cdots, n)$, $d_0 = 1$. Now we start an arithmetical argument.

Lemma 8. *For any $\sigma \in \{0, 1, \cdots, s\}$, we have*

$$d_n^{s-\sigma} P_{\sigma,n}(z) \in \mathbf{Z}[z]. \tag{12}$$

Proof. Fix n and j. Setting $F_\sigma(t)$ as follows, we have by decomposition into partial fractions:

$$F_\sigma(t) = \frac{(t-n\sigma)_n}{(t)_{n+1}}(t+j) = 1 + \sum_{\substack{p=0\\p\neq j}}^{n} \frac{(j-p)f_{p,\sigma}}{t+p}.$$

Similarly, by noting $H(t)$ as below, we have:

$$H(t) = \frac{n!}{(t)_{n+1}}(t+j) = \sum_{\substack{p=0\\p\neq j}}^{n} \frac{(j-p)h_p}{t+p}.$$

Here, we denote by $f_{p,\sigma}$ and by h_p:

$$f_{p,\sigma} = \frac{(-p-n\sigma)_n}{\prod_{h=0,h\neq p}^{n}(-p+h)} = \frac{(-1)^n((\sigma-1)n+p+1)_n}{(-1)^p p!(n-p)!}$$

$$= (-1)^{n-p}\binom{n\sigma+p}{n}\binom{n}{p},$$

$$h_p = \frac{n!}{\prod_{h=0,h\neq p}^{n}(-p+h)} = \frac{(-1)^p \cdot n!}{p!(n-p)!} = (-1)^p\binom{n}{p}.$$

For an integer $\lambda \geqslant 0$, let $\delta_{0,0} = 1$ and $\delta_{0,\lambda} = 0$ if $\lambda > 0$. Then we have

$$(D_\lambda F_\sigma(t))|_{t=-j} = \delta_{0,\lambda} + \sum_{p=0,p\neq j}^{n} (-1)^\lambda \frac{(j-p)f_{p,\sigma}}{(p-j)^{\lambda+1}},$$

$$(D_\lambda H(t))|_{t=-j} = \sum_{p=0,p\neq j}^{n} (-1)^\lambda \frac{(j-p)h_p}{(p-j)^{\lambda+1}}.$$

Thus for any integer $0 \leqslant \lambda \in \mathbf{Z}$, we have shown for $d_n = \mathrm{lcm}(1, 2, \cdots, n)$:

$$d_n^\lambda (D_\lambda F_\sigma(t))|_{t=-j} \in \mathbf{Z}, \quad d_n^\lambda (D_\lambda H(t))|_{t=-j} \in \mathbf{Z}.$$

For $\mu \in \mathbf{N}^s$ with $\mu_1 + \cdots + \mu_s = s - \sigma$, we obtain by Leibniz formula:

$$D_{s-\sigma}(R(t)(t+j)^s) = \sum_{\mu} (D_{\mu_1}F_1)\cdots(D_{\mu_r}F_r)(D_{\mu_{r+1}}H)\cdots(D_{\mu_s}H).$$

We get $d_n^{s-\sigma} c_{\sigma,j,n} \in \mathbf{Z}$ namely $d_n^{s-\sigma} P_{\sigma,n}(z) \in \mathbf{Z}[z]$. □

Recall $\alpha = p/q \in \mathbf{Q}$, $0 < |\alpha| < 1$. We set

$$p_{\sigma,n} = d_n^s p^n P_{\sigma,n}(q/p), \quad \sigma \in \{0, \cdots, s\}, \tag{13}$$

$$\ell_n = d_n^s p^n N_n(q/p) = p_{0,n} + \sum_{\sigma=1}^{a} p_{\sigma,n} \mathrm{Li}_\sigma(\alpha).$$

Putting

$$A = e^s |p| \varphi_{r,s}(1/\alpha), \tag{14}$$

$$B = e^s |q| 2^{s+r+1} r^r, \tag{15}$$

we have $A > 0$ and $B > 1$. By writing $[B^n]$ the largest integer $\leqslant B^n$ (floor function), we set

$$Q_n = [B^n] + 1. \tag{16}$$

Lemma 6 shows that the quantity A is small enough. On the other hand, by an explicit version of Prime Number Theorem in [8], we have estimates for d_n.

Theorem E (Rosser and Schoenfeld)**.** *Let* $R = \frac{515}{(\sqrt{546} - \sqrt{322})^2}$ *and*

$$\varepsilon(n) = (\log n)^{1/2} \exp\{-\sqrt{(\log n)/R}\}.$$

Then for any $n \geqslant 2$, we have

$$n\{1 - \varepsilon(n)\} \leqslant \log d_n \leqslant n\{1 + \varepsilon(n)\}. \tag{17}$$

3.2. Parameters. Theorem E gives us together with (10), (11) and (17):

$$\log |p_{\sigma,n}|^{1/n} \leqslant \log B + o(1), \tag{18}$$

$$\log |\ell_n|^{1/n} = \log A + o(1), \tag{19}$$

on the other hand, by definition of $[\cdot]$, we have

$$Q_{n+1} = Q_n^{1+o(1)}, \tag{20}$$

$$|p_{\sigma,n}| \leqslant Q_n^{1+o(1)}. \tag{21}$$

Moreover, Lemma 8 says $d_n^{s-\sigma} P_{\sigma,n}(z) \in \mathbf{Z}[z]$, hence by definition of $p_{\sigma,n}$ in (13), we have

$$p_{\sigma,n} = d_n^s p^n P_{\sigma,n}(q/p) = d_n^\sigma \times d_n^{s-\sigma} p^n P_{\sigma,n}(q/p)$$

namely

$$\frac{p_{\sigma,n}}{d_n^\sigma} \in \mathbf{Z}. \tag{22}$$

Here, $o(1)$ is defined with respect to n.

By the fact (22), the Fischler-Zudilin criterion gives a contribution which allows us to obtain the final lower bound below in the forthcoming part of our proof:

$$\tau + \gamma_1 + \cdots + \gamma_M = \tau + \sum_{\sigma=1}^{M} \frac{\sigma}{\log B}.$$

If we use Nesterenko's criterion, then the final lower bound is τ.

We now choose the parameter τ.

Proposition 9. *Put*

$$\tau = -\frac{\log A}{\log B}. \tag{23}$$

Then we have

$$|\ell_n| = Q_n^{-\tau+o(1)} \tag{24}$$

where $o(1)$ is defined with respect to n.

Proof. By (19) and by definition of Q_n, we have

$$\frac{\log|\ell_n|}{\log([B^n]+1)} \geqslant \frac{\log|\ell_n|}{(n+1)\log B} = \left(1 - \frac{1}{n+1}\right)\frac{(1/n)\log|\ell_n|}{\log B} = \frac{\log A}{\log B} + o(1).$$

On the other hand, we see:

$$\frac{\log|\ell_n|}{\log([B^n]+1)} \leqslant \frac{\log|\ell_n|}{n\log B} = \frac{\log A}{\log B} + o(1).$$

The definition of τ yields the statement. □

Proposition 10. *Put*

$$\gamma_\sigma = \frac{\sigma}{\log B}. \tag{25}$$

Then we have

$$d_n^\sigma = Q_n^{\gamma_\sigma + o(1)}. \tag{26}$$

Proof. Since $\varepsilon(n) = \dfrac{\sqrt{\log n}}{e^{\sqrt{(\log n)/R}}} = o(1)$, we have

$$\frac{\sigma \log d_n}{\log([B^n]+1)} \geqslant \frac{\sigma\{1-\varepsilon(n)\}n}{(n+1)\log B} = \sigma\left(1 - \frac{1}{n+1}\right)\frac{1-\varepsilon(n)}{\log B} = \frac{\sigma}{\log B} + o(1).$$

Similarly,

$$\frac{\sigma \log d_n}{\log([B^n]+1)} \leqslant \frac{\sigma\{1+\varepsilon(n)\}n}{n\log B} = \frac{\sigma}{\log B} + o(1).$$

Therefore

$$\gamma_\sigma = \frac{\sigma}{\log B} \Rightarrow \gamma_\sigma + o(1) = \frac{\log d_n^\sigma}{\log([B^n]+1)}$$

which is the statement. □

Proposition 11. *Let* $d_n = \mathrm{lcm}(1, 2, \cdots, n)$. *Then*

$$d_n^\sigma \textit{ divides } d_n^{\sigma+1} \textit{ for any } n \geqslant 1,$$

$$d_n^{\sigma'-\sigma} \textit{ divides } d_{n+1}^{\sigma'-\sigma} \textit{ for any } n \geqslant 1,\ 0 \leqslant \sigma < \sigma' \leqslant r.$$

Proof. Obvious. □

3.3. Choice of r. We start our proof of Theorem 2. We may suppose $M \geqslant 3$ since when $M = 1$ and $M = 2$, the statement is trivial.

By writing $[a]$ the largest integer part $\leqslant a$ (floor function), we recall the choice of $r \in \mathbf{Z}$ in (8) given by

$$r = \max\left\{1, \left[\frac{M}{(\log\max\{3, M\})^\rho}\right]\right\}$$

with $\rho > 0$ arbitrarily chosen and fixed, $[a]$ the largest integer part $\leqslant a$ (floor function).

Then thanks to (18) (19) (20) (21) (22) (24) (26) with Proposition 11, the hypothesis of Theorem C are satisfied with respect to $\delta_{i,n} = d_n^\sigma$ (here, we understand $i = \sigma$). The relations (14) (15) (23) (25) yield

$$M \geqslant \tau + \gamma_1 + \cdots \gamma_M \geqslant \frac{-M - \log|p| + (M-r)\log r + \frac{M(M+1)}{2}}{M + \log|q| + (M+r+1)\log 2 + r\log r}.$$

Hence the conclusion follows.

4. Proof of Theorem 1.

Our Theorem 1 is a consequence of Theorem D whenever

$$AB < \lim_{n\to\infty} \inf(\gcd(\delta_n, \delta_{n+1}))^{1/n} \tag{27}$$

where δ_n is a common divisor of $p_{1,n}, \cdots, p_{s,n}$. In our case, we take $\delta_n = d_n$.

Choose $r \in \mathbf{Z}$, $1 \leqslant r < s$ for example with $\rho = 3/2$:

$$r = \max\left\{1, \left[\frac{s}{(\log \max\{3, s\})^{3/2}}\right]\right\}. \tag{28}$$

Our construction of the sequence $p_{\sigma,n}$ satisfies the hypothesis (5) (6) (7). Define the function

$$T(s, r, p, q) := s \log r + 1 - 2s - (s + r + 1) \log 2 \\ - 2r \log r - \log |p| - \log |q|.$$

Let r be as chosen in (28). The function $T(s, r, 49, 50)$ is discontinuous (because of the floor function in r), however it has a zero at $s = 355.99\ldots$ and $T(s, r, 49, 50)$ is increasing when $s > 300$. Then for all $1 \leqslant |p| \leqslant 49$ and $2 \leqslant |q| \leqslant 50$, we have $T(s, r, p, q) > 0$ if $s \geqslant 356$. For such p, q, the condition (27) is verified by our definition of A, B.

The statement of Theorem 1 is therefore achieved.

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