Zeta functions of generalized permutations with application to their factorization formulas

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Abstract: We obtain a determinant expression of the zeta function of a generalized permutation over a finite set. As a corollary we prove the functional equation for the zeta function. In view of absolute mathematics, this is an extension from $GL(n, \mathbf{F}_1)$ to $GL(n, \mathbf{F}_{1^m})$, where $\mathbf{F}_1$ and $\mathbf{F}_{1^m}$ denote the imaginary objects "the field of one element" and "its extension of degree m", respectively. As application we obtain a certain product formula for the zeta function, which is analogous to the factorization of the Dedekind zeta function into a product of Dirichlet L-functions for an abelian extention.

Key words: Zeta functions; the field with one element; absolute mathematics; generalized permutation groups.

1. Introduction. Let

$$(1)\qquad \zeta_\sigma(s) = \exp\left(\sum_{m=1}^{\infty} \frac{|\mathrm{Fix}(\sigma^m)|}{m} e^{-ms}\right)$$

be the zeta function of the $\mathbf{Z}$-dynamical system generated by a permutation $\sigma \in S_n$, where S_n denotes the symmetric group over $X_n = \{1, \ldots, n\}$. We see that $\zeta_\sigma(s)$ is determined by the conjugacy class of σ in S_n. By Proposition 1 below, it is also expressed by the Euler product over the set $\mathrm{Cyc}(\sigma)$ of primitive cycles of σ:

$$\zeta_\sigma(s) = \prod_{p \in \mathrm{Cyc}(\sigma)} (1 - N(p)^{-s})^{-1},$$

where $N(p) = e^{l(p)}$ with $l = l(p)$ being the length of a primitive cycle

$$(2)\qquad p:\ i \mapsto \sigma(i) \mapsto \sigma^2(i) \mapsto \cdots \mapsto \sigma^l(i) = i$$

for some $i \in \{1, \ldots, n\}$.

In our previous paper [3], we gave a proof of the determinant expression

$$(3)\qquad \zeta_\sigma(s) = \det(I - M(\sigma)e^{-s})^{-1},$$

which enables us to obtain the functional equation of $\zeta_\sigma(s)$.

Our first goal is to generalize such properties to the case of generalized permutations. Consequently we generalize $\zeta_\sigma(s)$ to $L_\sigma(s, \chi)$ with χ a function over the set of cycles. As application we obtain a certain product formula for the zeta function, which is analogous to the factorization of the Dedekind zeta function into a product of Dirichlet L-functions in the case of an abelian extention.

We first briefly recall the definitions and settings on the generalized symmetric groups following the notation in [1].

Let ξ be a primitive m-th root of unity, and $\boldsymbol{\mu}_m$ be the multiplicative group of m-th roots of unity. The generalized permutation group W_n^m is the Wreath product of $\boldsymbol{\mu}_m$ by S_n:

$$1 \to (\boldsymbol{\mu}_m)^n \to W_n^m \to S_n \to 1.$$

It is also expressed as the group of permutations τ of the set

$$(4)\qquad X_{n,m} := \{\xi^k i \mid i = 1, \ldots, n,\ k = 0, 1, \ldots, m-1\}$$

such that $\tau(\xi^k i) = \xi^k \tau(i)$ for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, m-1$. The order of W_n^m is $m^n n!$. The group W_n^m has the following presentation ([2]):

$$\begin{aligned} W_n^m = \langle r_1, \ldots, r_{n-1}, w_1, \ldots, w_n\ :\ & \\ r_i^2 = (r_i r_{i+1})^3 = (r_i r_j)^2 = e,\ & \text{if } |i-j| \geq 2, \\ w_i^m = e,\ w_i w_j = w_j w_i,\ & r_i w_i = w_{i+1} r_i, \\ r_i w_j = w_j r_i,\ & \text{if } j \neq i, i+1\rangle. \end{aligned}$$

We may identify r_i $(i = 1, \ldots, n-1)$ with the transposition $(i, i+1)$ and therefore the symmetric group is

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$$S_n = \langle r_1, \ldots, r_{n-1} \rangle.$$

The elements w_i may be identified with the mapping $X_{n,m} \longrightarrow X_{n,m}$ defined by

$$w_i(\xi^k j) = \begin{cases} \xi^{k+1} j & (j = i) \\ \xi^k j & (j \neq i) \end{cases}.$$

An element $\tau \in W_n^m$ is determined by the images from the base space X_n, which is embedded in X_{nm} with $k = 0$ in (4). Namely, it can be written as

$$\text{(5)} \qquad \tau = \begin{pmatrix} 1 & 2 & \cdots & n \\ \xi^{s_1}\sigma(1) & \xi^{s_2}\sigma(2) & \cdots & \xi^{s_n}\sigma(n) \end{pmatrix} = \sigma \prod_{i=1}^{n} w_i^{s_i} \in W_n^m$$

with $\sigma \in S_n$ and $s_j \in \{0, 1, 2, \ldots, m-1\}$. Denote by M the canonical representation $M : W_n^m \to GL_n(\mathbf{C})$ of W_n^m defined by $M(\tau) = (\xi^{s_i}\delta_{\sigma(i),j})_{i,j=1,\ldots,n}$.

We define a function $\chi = \chi_\tau$ on the set of primitive cycles p of σ given by (2) as

$$\text{(6)} \qquad \begin{array}{cccc} \chi : & \mathrm{Cyc}(\sigma) & \to & \mathbf{C}^\times \\ & p & \mapsto & \xi^{\int_p \tau} \end{array}$$

with $\int_p \tau = \sum_{i \in p} s_i$. We also define the attached L-function as

$$\text{(7)} \qquad L_\sigma(s, \chi) = \prod_{p \in \mathrm{Cyc}(\sigma)} (1 - \chi(p) N(p)^{-s})^{-1}.$$

Our first main result is the determinant expression of $L_\sigma(s, \chi)$ described in Theorem 1 below. It is a natural extension of (3) from the viewpoint of absolute mathematics, because the symmetric group is interpreted as $S_n = GL(n, \mathbf{F}_1)$, and the generalized permutation group is $W_n^m = GL(n, \mathbf{F}_{1^m})$. As corollaries of Theorem 1, we obtain the functional equation and the tensor structure of $L_\sigma(s, \chi)$.

Finally in the last section we reach a factorization formula which is an analog of the decomposition of the Dedekind zeta function of an abelian extension into Hecke L-functions.

2. Determinant expression. In our previous paper [3] we proved the following proposition.

Proposition 1. *Let X and Y be finite sets. Put $|X| = n$. For $\sigma \in S_n$, the following properties hold.*

(i) *$\zeta_\sigma(s)$ has a determinant expression*

$$\zeta_\sigma(s) = \det(1 - M_0(\sigma)e^{-s})^{-1},$$

where $M_0(\sigma) = (\delta_{\sigma(i),j})_{i,j=1,\ldots,n}$ is the matrix representation $M_0 : S_n \to GL_n(\mathbf{C})$.

(ii) *$\zeta_\sigma(s)$ satisfies an analog of the Riemann hypothesis: $\zeta_\sigma(s) = \infty$ implies $\mathrm{Re}(s) = 0$.*

(iii) *$\zeta_\sigma(s)$ satisfies the functional equation*

$$\zeta_\sigma(-s) = \zeta_\sigma(s)(-1)^n \operatorname{sgn}(\sigma)e^{-ns}.$$

(iv) *$\zeta_\sigma(s)$ has the Euler product*

$$\zeta_\sigma(s) = \prod_{p \in \mathrm{Cyc}(\sigma)} (1 - N(p)^{-s})^{-1}.$$

(v) *The singularities of $\zeta_\sigma(s)$ satisfy an additive structure under the tensor product. Namely, the sum of a pole of $\zeta_\sigma(s)$ for $\sigma \in \mathrm{Aut}(X)$ and a pole of $\zeta_\tau(s)$ for $\tau \in \mathrm{Aut}(Y)$ is a pole of $\zeta_{\sigma\otimes\tau}(s)$, and all poles of $\zeta_{\sigma\otimes\tau}(s)$ are given by this way. Here for $\sigma \in \mathrm{Aut}(X)$ and $\tau \in \mathrm{Aut}(Y)$, we denote their tensor product by $\sigma \otimes \tau \in \mathrm{Aut}(X \times Y)$.*

(vi) *The Laurent expansion of $\zeta_\sigma(s)$ around $s = 0$ is given as follows:*

$$\zeta_\sigma(s) = s^{-m}c(\sigma)^{-1} + O(s^{-m+1}),$$

where m is the multiplicity of the eigenvalue 1 of $M_0(\sigma)$ and $c(\sigma) = \prod_{p \in \mathrm{Cyc}(\sigma)} l(p)$.

In this section we prove a generalization of this proposition to $L_\sigma(s, \chi)$.

Theorem 1. *Let X be a finite set with $|X| = n$, and $\xi \in \mathbf{C}$ be a primitive m-th root of unity. For a generalized permutation $\tau \in W_n^m$ with a decomposition given by (5), the L-function $L_\sigma(s, \chi)$ satisfies the determinant expression*

$$\text{(8)} \qquad L_\sigma(s, \chi) = \det(1 - M(\tau)e^{-s})^{-1}.$$

Note that the matrix $M(\tau)$ is not uniquely determined for each given χ. In other words, more than one τ's (or s_i's) may possibly correspond to the same χ. The determinant in (8), however, is well-defined for each χ not depending on the choice of τ or s_i's.

Proof of Theorem 1. We put the decomposition of a permutation σ into cyclic permutations as

$$\begin{aligned} \sigma &= \sigma_1 \cdots \sigma_r \\ &= (i_1, \ldots, i_{l(1)})(i_{l(1)+1}, \ldots, i_{l(1)+l(2)}) \\ &\qquad \cdots (i_{l(1)+\cdots+l(r-1)+1}, \ldots, i_n). \end{aligned}$$

Let $\pi \in S_n$ be the permutation such that $\pi(k) = i_k$ for $k = 1, 2, 3, \ldots n$. Then

$$\begin{aligned} \pi^{-1}\sigma\pi = (1 \cdots l(1))(l(1) + 1 \cdots l(1) + l(2)) \\ \cdots (l(1) + \cdots + l(r-1) + 1 \cdots n). \end{aligned}$$

Hence

$$M(\pi)^{-1}M(\tau)M(\pi) = \mathrm{diag}(C_{l(1)}, C_{l(2)}, \cdots, C_{l(r)})$$

with

$$C_{l(k)} \in \begin{pmatrix} 0 & \boldsymbol{\mu}_m & & \\ & \ddots & \ddots & \\ & & \ddots & \boldsymbol{\mu}_m \\ \boldsymbol{\mu}_m & & & 0 \end{pmatrix}$$

being a generalized cyclic permutation matrix of size $l(k)$. We define integers $t_1, \ldots, t_n \in \{0, 1, 2, \ldots, m-1\}$ by

$$C_{l(k)} = \begin{pmatrix} 0 & \xi^{t_{l(k-1)+1}} & & \\ & \ddots & \ddots & \\ & & \ddots & \xi^{t_{l(k-1)+l(k)-1}} \\ \xi^{t_{l(k-1)+l(k)}} & & & 0 \end{pmatrix}$$

with $l(0) = 0$ by convention. Note that $\{t_j\}$ is a reordered sequence of $\{s_j\}$. Since a cyclic permutation is corresponding to a cycle, we may write

$$\chi(C_{l(k)}) = \prod_{j=1}^{l(k)} \xi^{t_{l(k-1)+j}}$$

by taking the definition (6) into consideration. Then

$$\begin{aligned} &\det(1 - M(\tau)e^{-s}) \\ &= \det(1 - M(\pi)^{-1}M(\tau)M(\pi)e^{-s}) \\ &= \prod_{j=1}^{r} \det(I_{l(j)} - C_{l(j)}e^{-s}) \\ &= \prod_{j=1}^{r} (1 - \chi(C_{l(j)})e^{-l(j)s}), \end{aligned}$$

where the last identity is deduced by the following lemma. It holds that

$$\det(1 - M(\tau)e^{-s}) = \prod_{p \in \mathrm{Cyc}(\sigma)} (1 - \chi(p)N(p)^{-s}).$$

Theorem follows from the definiton (7). □

Lemma 1. *Let*

$$C_l = \begin{pmatrix} 0 & \xi^{t_1} & & \\ & \ddots & \ddots & \\ & & \ddots & \xi^{t_{l-1}} \\ \xi^{t_l} & & & 0 \end{pmatrix}$$

be a generalized permutation matrix. Put

$$\chi(C_l) = \prod_{j=1}^{l} \xi^{t_j}.$$

The following identity hold:

$$\det(I_l - C_l u) = 1 - \chi(C_l)u^l.$$

Proof.

$$\begin{aligned} &\det(I_l - C_l u) \\ &= \det \begin{pmatrix} 1 & -\xi^{t_1}u & & \\ & \ddots & \ddots & \\ & & \ddots & -\xi^{t_{l-1}}u \\ -\xi^{t_l}u & & & 1 \end{pmatrix} \\ &= 1 \cdot \begin{vmatrix} 1 & -\xi^{t_1}u & & \\ & 1 & \ddots & \\ & & \ddots & -\xi^{t_{l-2}}u \\ & & & 1 \end{vmatrix} \\ &\quad + (-1)^{l+1}(-\xi^{t_l}u) \begin{vmatrix} -\xi^{t_1}u & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & -\xi^{t_{l-1}}u \end{vmatrix} \\ &= 1 + (-1)^{l+1}(-u)(-u)^{l-1}\chi(C_l) \\ &= 1 - \chi(C_l)u^l. \end{aligned}$$

□

Corollary 1 (Functional equation)**.** *For a generalized permutation $\tau \in W_n^m$ with a decomposition given by (5), the L-function $L_\sigma(s, \chi)$ satisfies the functional equation*

$$L_\sigma(-s, \chi) = (-1)^n \det M(\tau)^{-1} e^{-ns} L_\sigma(s, \bar{\chi})$$

where $\overline{\chi}$ is the complex conjugation of χ which is given by replacing ξ with $\overline{\xi}$.

Proof. By Theorem 1, it follows that

$$\begin{aligned} &L_\sigma(-s, \chi) \\ &= \det(1 - M(\tau)e^{s})^{-1} \\ &= \det((-M(\tau)e^{s})(1 - M(\tau)^{-1}e^{-s}))^{-1} \\ &= (-1)^n(\det M(\tau))^{-1}e^{-ns}\det(1 - M(\tau)^{-1}e^{-s})^{-1}. \end{aligned}$$

□

The determinant expression in Theorem 1 also gives the tensor structure of L-functions in the following sense.

Let ξ_k be a primitive m_k-th root of unity for $k=1,2$. For generalized permutations $\tau_1 \in W_{n_1}^{m_1}$ over $X_{n_1}=\{1,\dots,n_1\}$ and $\tau_2 \in W_{n_2}^{m_2}$ over $X_{n_2}=\{1,\dots,n_2\}$ with their decomposition given by

$$\tau_k = \begin{pmatrix} 1 & 2 & \cdots & n_k \\ \xi_k^{s_{k,1}}\sigma_k(1) & \xi_k^{s_{k,2}}\sigma_k(2) & \cdots & \xi_k^{s_{k,n}}\sigma_k(n) \end{pmatrix} = \sigma_k \prod_{i=1}^{n_k} w_{k,i}^{s_{k,i}} \in W_{n_k}^{m_k}, \tag{9}$$

we define their tensor product $\tau_1 \otimes \tau_2 \in W_{n_1 n_2}^{m_1 m_2}$ as follows. As we saw in the notation (5), any element in $W_{n_1 n_2}^{m_1 m_2}$ is determined if we give the image of every element in the base space $X_{n_1 n_2} \cong X_{n_1} \times X_{n_2}$, which is given by

$$\begin{array}{rccc} \tau_1 \otimes \tau_2: & X_{n_1} \times X_{n_2} & \rightarrow & X_{n_1,m_1} \times X_{n_2,m_2} \\ & (i,j) & \mapsto & (\xi_1^{s_{1,i}}\sigma_1(i),\ \xi_2^{s_{2,j}}\sigma_2(j)) \\ & & \hookrightarrow & X_{n_1 n_2, m_1 m_2} \\ & & \mapsto & \xi^{m_2 s_{1,j} + m_1 s_{2,j}}(\sigma_1(i),\ \sigma_2(j)) \end{array}$$

with ξ a primitive $m_1 m_2$-th root of unity. In other words, if we identify $\tau_k \in W_{n_k}^{m_k}$ as the linear map $\tau_k : \mathbf{C}^{n_k} \to \mathbf{C}^{n_k}$ introduced by the representation M, the tensor product

$$\tau_1 \otimes \tau_2 : \ \mathbf{C}^{n_1} \otimes \mathbf{C}^{n_2} \to \mathbf{C}^{n_1} \otimes \mathbf{C}^{n_2}$$

is defined by the usual tensor product of linear maps with the representation matrix given by the Kronecker tensor product $M(\tau_1) \otimes M(\tau_2)$ of matrices.

In the following corollary, we define $\chi_1 = \chi_{\tau_1}$, $\chi_2 = \chi_{\tau_2}$, and $\chi_1 \otimes \chi_2 := \chi_{\tau_1 \otimes \tau_2}$.

Corollary 2 (Tensor structure). *The singularities of $L_\sigma(s,\chi)$ satisfy an additive structure under the tensor product. Namely, the sum of a pole of $L_1(s) := L_{\sigma_1}(s,\chi_1)$ and a pole of $L_2(s) := L_{\sigma_2}(s,\chi_2)$ is a pole of $L_{\sigma_1 \otimes \sigma_2}(s, \chi_1 \otimes \chi_2)$, and all poles of $L_{\sigma_1 \otimes \sigma_2}(s,\chi_1 \otimes \chi_2)$ are given in this way.*

Proof. By Theorem 1,

$$\begin{aligned} &L_{\sigma_1 \otimes \sigma_2}(s, \chi_1 \otimes \chi_2) \\ &\quad = \det(1 - M(\tau_1 \otimes \tau_2) e^{-s})^{-1} \\ &\quad = \det(1 - M(\tau_1) \otimes M(\tau_2) e^{-s})^{-1}. \end{aligned}$$

We put the eigenvalues of $M(\tau_1)$ and $M(\tau_2)$ as α_j $(j = 1,\dots,n_1)$ and β_k $(k = 1,2,\dots,n_2)$, respectively. We see from Theorem 1 that the poles of $L_{\sigma_1}(s,\chi_1)$ and $L_{\sigma_2}(s,\chi_2)$ are given by $s \equiv \log \alpha_j$ and $s \equiv \log \beta_k \pmod{2\pi i \mathbf{Z}}$. Thus the set of poles of $L_{\sigma_1 \otimes \sigma_2}(s, \chi_1 \otimes \chi_2)$ is given by

$$\{\log \alpha_j \beta_k \mod 2\pi i \mathbf{Z} \mid 1 \le j \le n_1,\ 1 \le k \le n_2\}.$$

The result follows from

$$\log \alpha_j \beta_k \equiv \log \alpha_j + \log \beta_k \pmod{2\pi i \mathbf{Z}}.$$

□

Theorem 1 also describes the order of the L-function at $s=0$ as follows.

Corollary 3. *The Laurent expansion of $L_\sigma(s,\chi)$ around $s=0$ is given as follows:*

$$L_\sigma(s,\chi) = s^{-K} c(\tau) + O(s^{-K+1}),$$

where K is the multiplicity of the eigenvalue 1 of $M(\tau)$ and

$$c(\tau) = \prod_{\substack{p \in \mathrm{Cyc}(\sigma) \\ \chi(p)=1}} (l(p))^{-1} \times \prod_{\substack{p \in \mathrm{Cyc}(\sigma) \\ \chi(p) \neq 1}} (1 - \chi(p))^{-1}.$$

Moreover, K is equal to the number of primitive cycles p of σ such that $\chi(p) = 1$.

Proof. By Theorem 1, we have

$$\begin{aligned} L_\sigma(s,\chi) &= \det(1 - M(\tau) e^{-s})^{-1} \\ &= \left((1 - e^{-s})^K \prod_{\alpha \neq 1} (1 - \alpha e^{-s}) \right)^{-1}, \end{aligned}$$

where in the last product α runs through the eigenvalues of $M(\tau)$ such that $\alpha \neq 1$. Hence $L_\sigma(s,\chi)$ has a pole of order K at $s=0$. The leading coefficient is calculated from (iv):

$$\begin{aligned} &\prod_{p \in \mathrm{Cyc}(\sigma)} (1 - \chi(p) N(p)^{-s})^{-1} \\ &\quad = \prod_{p \in \mathrm{Cyc}(\sigma)} (1 - \chi(p) + \chi(p) l(p) s + O(s^2))^{-1} \\ &\quad = s^{-K} \prod_{\substack{p \in \mathrm{Cyc}(\sigma) \\ \chi(p)=1}} (l(p))^{-1} \\ &\qquad \times \prod_{\substack{p \in \mathrm{Cyc}(\sigma) \\ \chi(p) \neq 1}} (1 - \chi(p) + \chi(p) l(p) s)^{-1} + O(s^{-K+1}). \end{aligned}$$

□

3. Factorization formulas. It is classical that for any finite abelian extention K/k of algebraic number fields of finite degree, the Dedekind zeta function $\zeta_K(s)$ is decomposed into the product of Dirichlet L-functions over Dirichlet characters:

$$\zeta_K(s) = \prod_\chi L_k(s,\chi). \tag{10}$$

In this section we obtain an analog of this phenomenon by restricting ourselves to the case when the function χ has the form

$$\chi(p) = \theta^{l(p)} \qquad (\forall p \in \mathrm{Cyc}(\sigma))$$

for some fixed $\theta \in \boldsymbol{\mu}_m$. Namely,

$$L_\sigma(s,\chi) = \prod_{p\in\mathrm{Cyc}(\sigma)} (1-\theta^{l(p)}e^{-l(p)s})^{-1} = \zeta_\sigma(s-\log\theta).$$

For $\theta = \exp(\frac{2\pi i}{m})$ $(m \in \mathbf{N})$, we denote $\chi = \chi_m$. The following factorization formula is analogous to (10).

Theorem 2. *Let $\sigma \in S_n$, and $\tau = \sigma\prod_{i=1}^n w_i \in W_n^m$.*

Put $\tilde\sigma$ to be the permutation τ regarded as an element in S_{nm}. Then it holds for any $m \in \mathbf{N}$ that

$$\zeta_{\tilde\sigma}(s) = \prod_{b=0}^{m-1} L_\sigma(s,\chi_m^b).$$

Before proving this theorem, we set up some analogous notions on lifting and splitting by following the theory of extensions of number fields. Consider the following commutative diagram

$$\begin{array}{ccc} \widetilde X & \xrightarrow{\widetilde\sigma} & \widetilde X \\ f\downarrow & & \downarrow f \\ X & \xrightarrow{\sigma} & X, \end{array}$$

where $f : \widetilde X \to X$ is a surjective map of finite sets with $\sigma \in \mathrm{Aut}(X)$ and $\widetilde\sigma \in \mathrm{Aut}(\widetilde X)$. Then a primitive cycle $\mathfrak{p} \in \mathrm{Cyc}(\widetilde\sigma)$ is called *a lift* of $p \in \mathrm{Cyc}(\sigma)$ if and only if $f(\mathfrak{p}) = p$. The inverse image $f^{-1}(p)$ of $p \in \mathrm{Cyc}(\sigma)$ is a (not necessarily primitive) cycle of $\widetilde\sigma$, and it can be decomposed into the form $f^{-1}(p) = \sum_{i=1}^g \mathfrak{p}_i$ with each $\mathfrak{p}_i$ a lift of p. In this setting we say that p *remains primitive* if $g = 1$, and that p *splits* if $g \geq 2$. Moreover, when $|f^{-1}(x)| = m$ for all $x \in X$, it holds that $g \leq m$, and we say that p *splits completely* if $g = m$.

Proof of Theorem 2. We appeal to the cyclotomic equation

$$\prod_{b=0}^{k-1}(1-\zeta_k^b X) = 1 - X^k$$

with ζ_k a primitive k-th root of unity. By putting $X = e^{-l(p)s}$ and $k = \frac{m}{(m,l(p))}$, we have

$$\begin{aligned}
\prod_{b=0}^{m-1} & L_\sigma(s,\chi_m^b) \\
&= \prod_{b=0}^{m-1}\prod_{p\in\mathrm{Cyc}(\sigma)}(1-\zeta_m^{bl(p)}e^{-l(p)s})^{-1} \\
&= \prod_{p\in\mathrm{Cyc}(\sigma)}\prod_{b=0}^{m-1}(1-\zeta_m^{bl(p)}e^{-l(p)s})^{-1} \\
&= \prod_{p\in\mathrm{Cyc}(\sigma)}\prod_{b=0}^{\frac{m}{(m,l(p))}-1}\left(1-(\zeta_m^{l(p)})^b e^{-l(p)s}\right)^{-(m,l(p))} \\
&= \prod_{p\in\mathrm{Cyc}(\sigma)}\left(1-e^{-\frac{ml(p)}{(m,l(p))}s}\right)^{-(m,l(p))}.
\end{aligned}$$

It remains to prove that the lifts of $p \in \mathrm{Cyc}(\sigma)$ are $(m,l(p))$ primitive cycles of $\tilde\sigma$ which are of length $\frac{ml(p)}{(m,l(p))}$.

To see this, we use the expression (4). Let $\xi^k i \in X_{n,m}$ be a fixed point of $\tilde\sigma^j$. Then,

$$\begin{aligned}\tilde\sigma^j(\xi^k i) = \xi^k i &\iff \sigma^j(i) = i \quad\text{and}\quad \theta^j\xi^k = \xi^k \\ &\iff l(p)|j \quad\text{and}\quad m|j,\end{aligned}$$

where p is the primitive cycle to which $i \in X_n$ belongs. Thus the length of the orbit of $\xi^k i$ is equal to the least common multiple of $l(p)$ and m, which is $\frac{ml(p)}{(m,l(p))}$.

The number of elements belonging to $f^{-1}(p)$ in $\tilde X$ is $ml(p)$. Thus the number of lifts of p is $(m,l(p))$ with their length $\frac{ml(p)}{(m,l(p))}$.

$\square$

From the proof of Theorem 2, we have the following facts immediately.

Corollary 4. *Let σ be a permutation of X_n, and p be a primitive cycle which belongs to* $\mathrm{Cyc}(\sigma)$ *with $l = l(p)$ defined as in (2).*

In the lifted permutation

$$\tilde\sigma :\ X_{n,m} \to X_{n,m}$$

of $\sigma : X_n \to X_n$, it holds that

$$\begin{cases} p \text{ remains primitive} & \text{if } (l,m) = 1, \\ p \text{ splits} & \text{if } (l,m) > 1. \end{cases}$$

In the extreme case, p splits completely, if and only if $m|l$.

This is analogous to the decomposition law of prime ideals for finite extensions of number fields.

Example 1. $n = 5$, $\sigma = (1\ 2)(3\ 4\ 5)$. $\mathrm{Cyc}(\sigma)$ consists of two primitive cycles p_1 and p_2, where $l(p_1) = 2$ and $l(p_2) = 3$. Consider the covering with $m = 2$, that is, $\xi = -1$. The cycle p_1 splits completely, since there exist two cycles above p_1, which are $(1 \mapsto -2 \mapsto 1)$ and $(2 \mapsto -1 \mapsto 2)$. Thus we find that p_1 splits completely in the extension $X_{5,2}$ of X_5. This is the case with $(m, l) = (2, 2) = 2$, which satisfies $m|l$.

On the other hand, the cycle p_2 remains primitive, because $p_2 = (3 \mapsto 4 \mapsto 5 \mapsto 3)$ is lifted to only one cycle $(3 \mapsto -4 \mapsto 5 \mapsto -3 \mapsto 4 \mapsto -5 \mapsto 3)$ of length 6. This is the case with $(l, m) = (3, 2) = 1$.

Example 2. $n = 8$, $\sigma = (1\ 2)(3\ 4\ 5\ 6\ 7\ 8)$. $\mathrm{Cyc}(\sigma)$ consists of two primitive cycles p_1 and p_2, where $l(p_1) = 2$ and $l(p_2) = 6$. Consider the covering with $m = 4$, that is, $\xi = \sqrt{-1} = i$. Above the cycle p_1 there exist two cycles of length 4, which are $(1 \mapsto 2i \mapsto -1 \mapsto -2i \mapsto 1)$ and $(2 \mapsto i \mapsto -2 \mapsto -i \mapsto 2)$. We find that p_1 splits in the extension $X_{8,4}$ of X_8. This is the case with $(l, m) = (2, 4) = 2 > 1$. The other cycle p_2 also splits, because there exists two cycles of length 12 above p_2. This is the case with $(l, m) = (6, 4) = 2 > 1$.

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