On log surfaces

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Abstract: This paper is an announcement of the minimal model theory for log surfaces in all characteristics and contains some related results including a simplified proof of the Artin–Keel contraction theorem in the surface case.

Key words: Contraction theorem; algebraic spaces; Frobenius map; vanishing theorem; minimal model theory; algebraic surfaces.

1. Introduction. In this paper, we will work over an algebraically closed field k of characteristic zero or positive characteristic. Let us recall the definition of *log surfaces*.

Definition 1.1 (Log surfaces)**.** Let X be a normal algebraic surface and let Δ be a boundary $\mathbf{R}$-divisor on X such that $K_X + \Delta$ is $\mathbf{R}$-Cartier. Then the pair (X, Δ) is called a *log surface*. We recall that a *boundary* $\mathbf{R}$-divisor is an effective $\mathbf{R}$-divisor whose coefficients are less than or equal to one.

In the preprint [Ta1] together with [Fn], we have obtained the following theorem, that is, the minimal model theory for log surfaces.

Theorem 1.2 (cf. [Fn] and [Ta1])**.** *Let $\pi : X \to S$ be a projective morphism from a log surface (X, Δ) to a variety S over a field k of arbitrary characteristic. Assume that one of the following conditions holds:*
(A) *X is $\mathbf{Q}$-factorial, or*
(B) *(X, Δ) is log canonical.*
Then we can run the log minimal model program over S with respect to $K_X + \Delta$ and obtain a sequence of extremal contractions

$$(X, \Delta) = (X_0, \Delta_0) \xrightarrow{\varphi_0} (X_1, \Delta_1) \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{k-1}} (X_k, \Delta_k) = (X^*, \Delta^*)$$

over S such that
(1) (Minimal model) *$K_{X^*} + \Delta^*$ is semi-ample over S if $K_X + \Delta$ is pseudo-effective over S, or*
(2) (Mori fiber space) *there is a morphism $g : X^* \to C$ over S such that $-(K_{X^*} + \Delta^*)$ is g-ample, $\dim C < 2$, and the relative Picard number $\rho(X^*/C) = 1$, if $K_X + \Delta$ is not pseudo-effective over S.*

We note that, in Case (A)*, we do not assume that (X, Δ) is log canonical. We also note that X_i is $\mathbf{Q}$-factorial for every i in* Case (A) *and that (X_i, Δ_i) is log canonical for every i in* Case (B)*. Moreover, in both cases, if X has only rational singularities, then so does X_i by* Theorem 6.2.

More precisely, we prove the cone theorem, the contraction theorem, and the abundance theorem for $\mathbf{Q}$-factorial log surfaces and log canonical surfaces with no further restrictions.

Theorem 1.2 has been known in Case (B) when S is a point and Δ is a $\mathbf{Q}$-divisor (cf. [Ft3], [KK]). In [Fn], the first author obtained Theorem 1.2 in characteristic zero; there are many $\mathbf{Q}$-factorial surfaces (i.e. in Case (A)) which are not log canonical (i.e. in Case (B)).

In [Ta1], the second author establishes Theorem 1.2 in arbitrary positive characteristic. The arguments in [Fn] heavily depend on a Kodaira type vanishing theorem, which unfortunately fails in positive characteristic. The main part of discussion in [Ta1] in order to prove Theorem 1.2 is the Artin–Keel contraction theorem, which holds only in positive characteristic.

We will give a simplified proof of the Artin–Keel contraction theorem in Section 2, which is one of the main purposes of this paper.

Theorem 1.2 implies the following important corollary. For a more direct approach to Corollary 1.3, see Section 3.

Corollary 1.3. *Let (X, Δ) be a projective log surface such that Δ is a $\mathbf{Q}$-divisor. Assume that X is*

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$\mathbf{Q}$-factorial or (X,Δ) is log canonical. Then the log canonical ring

$$R(X,\Delta)=\bigoplus_{m\geq 0}H^0(X,\mathcal{O}_X(\llcorner m(K_X+\Delta)\lrcorner))$$

is a finitely generated k-algebra.

Remark 1.4. Let (X,Δ) be a log canonical surface and let $f:Y\to X$ be its minimal resolution with $K_Y+\Delta_Y=f^*(K_X+\Delta)$. Then (Y,Δ_Y) is a $\mathbf{Q}$-factorial log surface. The abundance theorem and the finite generation of log canonical rings for (X,Δ) follow from those of (Y,Δ_Y).

In Section 5, we discuss a key result on the *indecomposable curves of canonical type* for the proof of the abundance theorem for $\kappa=0$. The behavior of the indecomposable curves of canonical type varies in accordance with the cases: (i) $\mathrm{char}(k)=0$, (ii) $\mathrm{char}(k)>0$ with $k\neq\overline{\mathbf{F}}_p$, and (iii) $k=\overline{\mathbf{F}}_p$. Section 6 is devoted to the discussion of some relative vanishing theorems, which are elementary and hold in any characteristic. As applications, we give some results supplementary to the theory of algebraic surfaces in *arbitrary* characteristic.

For the details of the proofs, we refer the reader to [Fn] and [Ta1].

Notation. For an $\mathbf{R}$-divisor D on a normal surface X, we define the *ronud-up* $\ulcorner D\urcorner$, the *round-down* $\llcorner D\lrcorner$, and the *fractional part* $\{D\}$ of D. We note that $\sim_{\mathbf{R}}$ denotes the $\mathbf{R}$*-linear equivalence* of $\mathbf{R}$-divisors. Let D be a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor on a normal projective surface X. Then $\kappa(X,D)$ denotes the *Iitaka–Kodaira dimension* of D. Let k be a field. Then $\mathrm{char}(k)$ denotes the characteristic of k. Let A be an abelian variety defined over an algebraically closed field k. Then we denote the k-rational points of A by $A(k)$.

2. The Artin–Keel contraction theorem.

The following (see Theorem 2.1) is Keel's base point free theorem for algebraic surfaces (cf. [Ke, 0.2 Theorem]). Although his original result holds in *any* dimension, we only discuss it for surfaces here. The paper [Ke] attributes Theorem 2.1 to [A] even though it is not stated explicitly there. So, we call it the Artin–Keel contraction theorem in this paper. Theorem 2.1 will play crucial roles in the minimal model theory for log surfaces in positive characteristic. For the details, see [Ta1]. Note that Theorem 2.1 fails in characteristic zero by [Ke, 3.0 Theorem]. The minimal model theory for log surfaces in characteristic zero heavily depends on a Kodaira type vanishing theorem (cf. [Fn]). The second author discusses the X-method for klt surfaces in positive characteristic in [Ta2]. For a related topic, see also [CMM].

Theorem 2.1 (Artin, Keel)**.** *Let X be a complete normal algebraic surface defined over an algebraically closed filed k of positive characteristic and let H be a nef and big Cartier divisor on X. We set*

$$\mathcal{E}(H):=\{C\mid C \textit{ is a curve on } X \textit{ and } C\cdot H=0\}.$$

Then $\mathcal{E}(H)$ consists of finitely many irreducible curves on X. Assume that $H|_{\mathbf{E}(H)}$ is semi-ample where

$$\mathbf{E}(H)=\bigcup_{C\in\mathcal{E}(H)}C$$

with the reduced scheme structure. Then H is semi-ample. Therefore,

$$\Phi_{|mH|}:X\to Y$$

is a proper birational morphism onto a normal projective surface Y which contracts $\mathbf{E}(H)$ and is an isomorphism outside $\mathbf{E}(H)$ for a sufficiently large and divisible positive integer.

We give two different proofs of Theorem 2.1. Proof 1 depends on Artin's arguments. On the other hand, Proof 2 uses Fujita's vanishing theorem.

Proof 1. It is sufficient to prove that H is semi-ample. Let $f:Z\to X$ be a resolution of singularities. Then $\mathcal{E}(f^*H)$ consists of finitely many curves by the Hodge index theorem. Therefore, so does $\mathcal{E}(H)$. Note that H is semi-ample if and only if f^*H is semi-ample. We also note that $f^*H|_{\mathbf{E}(f^*H)}$ is semi-ample since so is $H|_{\mathbf{E}(H)}$. Thus, by replacing X and H with Z and f^*H, we may assume that X is a smooth projective surface. In this case, the intersection matrix of $\mathcal{E}(H)$ is negative definite by the Hodge index theorem.

By Artin's contraction theorem (see [B, Theorem 14.20]), there exists a morphism $g:X\to W$ where W is a normal complete two-dimensional algebraic space such that $g(\mathbf{E}(H))$ is a finite set of points of W and that $g|_{X\setminus\mathbf{E}(H)}:X\setminus\mathbf{E}(H)\to W\setminus g(\mathbf{E}(H))$ is an isomorphism.

By Artin (cf. [A, Lemma (2.10)] and [B, Step 1 in the proof of Theorem 14.21]), there exists an effective Cartier divisor E with $\mathrm{Supp}(E)=\mathbf{E}(H)$ such that for every effective divisor $D\geq E$ with $\mathrm{Supp}D=\mathbf{E}(H)$, the restriction map $\mathrm{Pic}(D)\to\mathrm{Pic}(E)$ is an isomorphism.

By replacing H with a multiple, we may assume that $H|_{\mathbf{E}(H)}$ is free. Therefore, $\mathcal{O}_{\mathbf{E}(H)}(H) \simeq \mathcal{O}_{\mathbf{E}(H)}$.

Let p be the characteristic of k and let r be a positive integer such that $q\mathbf{E}(H) \geq E$ where $q = p^r$. We consider the (ordinary) q-th Frobenius morphism $F : X \to X$. By pulling back the exact sequence $0 \to \mathcal{O}_X(H - \mathbf{E}(H)) \to \mathcal{O}_X(H) \to \mathcal{O}_{\mathbf{E}(H)} \to 0$ by F, we obtain the exact sequence $0 \to \mathcal{O}_X(qH - q\mathbf{E}(H)) \to \mathcal{O}_X(qH) \to \mathcal{O}_{q\mathbf{E}(H)}(qH) \to 0$. Therefore, $\mathcal{O}_{q\mathbf{E}(H)}(qH) \simeq \mathcal{O}_{q\mathbf{E}(H)}$. By the above argument, $\mathcal{O}_D(qH) \simeq \mathcal{O}_D$ for every effective divisor $D \geq E$ with $\mathrm{Supp}D = \mathbf{E}(H)$.

Let $w \in g(\mathbf{E}(H))$ be a point. Then, by the theorem of holomorphic functions (cf. [Kn, Theorem 3.1]), we have

$$(g_*\mathcal{O}_X(qH))^\wedge_w \simeq (g_*\mathcal{O}_X)^\wedge_w \simeq \widehat{\mathcal{O}_{W,w}}.$$

Therefore, $g_*\mathcal{O}_X(qH)$ is a line bundle on W. By considering the natural map

$$g^*g_*\mathcal{O}_X(qH) \to \mathcal{O}_X(qH),$$

we obtain that $g^*g_*\mathcal{O}_X(qH) \simeq \mathcal{O}_X(qH)$ since the intersection matrix of $\mathcal{E}(H)$ is negative definite. This means that $B := g_*qH$ is a Cartier divisor on W and $g^*B = qH$. By Nakai's criterion, B is ample on W. We note that Nakai's criterion holds for complete algebraic spaces. Therefore, H is semi-ample. □

Proof 2. It is sufficient to prove that H is semi-ample. By the same argument as in Proof 1, we may assume that X is smooth. We may further assume that $\mathcal{O}_{\mathbf{E}(H)}(H) \simeq \mathcal{O}_{\mathbf{E}(H)}$ by replacing H with a multiple. Since the intersection matrix of $\mathcal{E}(H)$ is negative definite, we can find an effective Cartier divisor D such that $\mathrm{Supp}D = \mathbf{E}(H)$ and that $-D \cdot C > 0$ for every $C \in \mathcal{E}(H)$.

Claim. *There exists a positive integer m such that $mH - D$ is ample.*

Proof of Claim. By Kodaira's lemma, we can find a positive integer k, an ample Cartier divisor A, and an effective divisor B such that $kH - D \sim A + B$. If $(kH - D) \cdot C \leq 0$ for some curve C, then $C \subset \mathrm{Supp}B$ and $C \notin \mathcal{E}(H)$. Therefore, if $m \gg k$, then $(mH - D) \cdot C > 0$ for every curve C on X. This implies $mH - D$ is ample since $mH - D$ is a big divisor. □

By replacing $mH - D$ with a multiple, we may assume that $mH - D$ is very ample and $H^1(X, \mathcal{O}_X(lH - D)) = 0$ for every $l \geq m$ by Fujita's vanishing theorem (see [Ft1, Theorem (1)] and [Ft2, (5.1) Theorem]). Let p be the characteristic of k and let r be a positive integer such that $q\mathbf{E}(H) \geq D$ where $q = p^r$. By the same argument as in Proof 1, $\mathcal{O}_{q\mathbf{E}(H)}(qH) \simeq \mathcal{O}_{q\mathbf{E}(H)}$. Therefore, $qH|_D \sim 0$. Without loss of generality, we may further assume that $q \geq m$. By the exact sequence $0 \to H^0(X, \mathcal{O}_X(qH - D)) \to H^0(X, \mathcal{O}_X(qH)) \to H^0(D, \mathcal{O}_D(qH)) \to 0$, $\mathrm{Bs}|qH| \cap \mathbf{E}(H) = \emptyset$ where $\mathrm{Bs}|qH|$ is the base locus of the linear system $|qH|$. Since $mH - D$ is ample with $\mathrm{Supp}D = \mathbf{E}(H)$, we obtain that H is semi-ample. □

Corollary 2.2. *Let X be a $\mathbf{Q}$-factorial projective surface defined over an algebraically closed field of positive characteristic. Let C be a curve on X such that $C \simeq \mathbf{P}^1$ and $C^2 < 0$. Then we can contract C to a $\mathbf{Q}$-factorial point.*

Sketch of the proof. Let H be a very ample Cartier divisor on X. We set $L = (-C^2)H + (H \cdot C)C$. Then L is nef and big. Note that $L|_C$ is semi-ample since $C \simeq \mathbf{P}^1$ and $L \cdot C = 0$. By applying Theorem 2.1 to L, we have a desired contraction morphism. □

Since $\mathrm{Pic}^0(V)(k)$ is a torsion group for any projective variety V defined over $k = \overline{\mathbf{F}}_p$, we obtain the following corollary.

Corollary 2.3 (cf. [Ke, 0.3 Corollary])**.** *Let X be a normal projective surface over $k = \overline{\mathbf{F}}_p$ and let D be a nef and big Cartier divisor on X. Then D is semi-ample.*

In [Ke, Section 3], Keel obtained an interesting example.

Proposition 2.4 (cf. [Ke, 3.0 Theorem])**.** *Let C be a smooth projective curve of genus $g \geq 2$ over an algebraically closed filed k. We consider $S = C \times C$. We set $L = \pi_1^*K_C + \Delta$ where $\Delta \subset S$ is the diagonal and $\pi_1 : S \to C$ is the first projection. Then L is semi-ample if and only if the characteristic of k is positive.*

By Proposition 2.4, we obtain the following interesting example.

Example 2.5. Let U be a nonempty Zariski open set of $\mathrm{Spec}\mathbf{Z}$ and let $X \to U$ be a smooth family of curves of genus $g \geq 2$. We set $Y = X \times_U X$. Let Δ be the image of the diagonal morphism $\Delta_{X/U} : X \to X \times_U X$. We set $M = p_1^*K_{X/U} + \Delta$ where $p_1 : Y = X \times_U X \to X$ is the first projection. Let $p \in U$ be any closed point. Then $M_p = M|_{Y_p}$ is semi-ample and big for every $p \in U$, where $\pi : Y \to U$ is the natural map and $Y_p = \pi^{-1}(p)$. On the other hand, M is not π-semi-ample.

3. char(k) = 0 vs. char(k) > 0. The following theorem is a special case of the abundance theorem for log surfaces. It is a key step toward showing the finite generation of log canonical rings (see Corollary 1.3).

Theorem 3.1. *Let (X, Δ) be a $\mathbf{Q}$-factorial projective log surface such that Δ is a $\mathbf{Q}$-divisor. Assume that $K_X + \Delta$ is nef and big. Then $K_X + \Delta$ is semi-ample.*

When $\mathrm{char}(k) = 0$, the proof of Theorem 3.1 heavily depends on a Kodaira type vanishing theorem and it is one of the hardest parts of [Fn]. Section 4 of [Fn] is devoted to the proof of Theorem 3.1. On the other hand, when $\mathrm{char}(k) > 0$, the proof of Theorem 3.1 is much simpler by Theorem 2.1. Therefore, in some sense, the minimal model theory of log surfaces is easier to treat in positive characteristic. In $\mathrm{char}(k) = 0$, it follows from the mixed Hodge theory of compact support cohomology groups. In $\mathrm{char}(k) > 0$, it uses Frobenius maps (see the proof of Theorem 2.1).

Sketch of the proof ($\mathrm{char}(k) > 0$). First, we set $\mathcal{E}(K_X + \Delta) := \{C \mid C$ is a curve on X and $C \cdot (K_X + \Delta) = 0\}$. Then $\mathcal{E}(K_X + \Delta)$ consists of finitely many irreducible curves on X by the Hodge index theorem. We take an irreducible curve $C \in \mathcal{E}(K_X + \Delta)$. Then $C^2 < 0$ by the Hodge index theorem. If $(K_X + C) \cdot C < 0$, then $C \simeq \mathbf{P}^1$ by adjunction and we can contract C to a point by Corollary 2.2. Therefore, we may assume that C is an irreducible component of $\llcorner \Delta \lrcorner$, $C \cap \mathrm{Supp}(\Delta - C) = \emptyset$, and $(K_X + \Delta) \cdot C = 0$ for every $C \in \mathcal{E}(K_X + \Delta)$. If $C \simeq \mathbf{P}^1$ for $C \in \mathcal{E}(K_X + \Delta)$, then it is obvious that $(K_X + \Delta)|_C$ is semi-ample. If $C \not\simeq \mathbf{P}^1$ for $C \in \mathcal{E}(K_X + \Delta)$, then we can also check that $(K_X + \Delta)|_C$ is semi-ample by adjunction. Therefore, by Theorem 2.1, we obtain that $K_X + \Delta$ is semi-ample. □

For the details of Theorem 3.1, see [Fn, Section 4] and [Ta1, Section 5].

Sketch of the proof of Corollary 1.3 ($\mathrm{char}(k) > 0$). If $\kappa(X, K_X + \Delta) \le 1$, then it is obvious that $R(X, \Delta)$ is a finitely generated k-algebra. So, we assume that $K_X + \Delta$ is big. If $K_X + \Delta$ is not nef, then we can find a curve C on X such that $(K_X + \Delta) \cdot C < 0$ and $C^2 < 0$. Therefore, $(K_X + C) \cdot C < 0$. By adjunction, $C \simeq \mathbf{P}^1$. By Corollary 2.2, we can contract C. After finitely many steps, we may assume that $K_X + \Delta$ is nef. By Theorem 3.1, $K_X + \Delta$ is semi-ample. Thus, $R(X, \Delta)$ is a finitely generated k-algebra. □

4. $k \neq \overline{\mathbf{F}}_p$ vs. $k = \overline{\mathbf{F}}_p$.

First, we note the following important result.

Theorem 4.1 (see, for example, [Ta1, Theorem 10.1])**.** *Let X be a normal surface defined over $\overline{\mathbf{F}}_p$. Then X is $\mathbf{Q}$-factorial.*

One of the key results for the minimal model theory of $\mathbf{Q}$-factorial log surfaces is as follows. It plays crucial roles in the proof of the non-vanishing theorem and the abundance theorem for $\mathbf{Q}$-factorial log surfaces. For details, see [Fn] and [Ta1].

Theorem 4.2 (cf. [Fn, Lemma 5.2] and [Ta1, Theorem 4.1])**.** *Assume that $k \neq \overline{\mathbf{F}}_p$. Let X be a $\mathbf{Q}$-factorial projective surface and let $f : Y \to X$ be a projective birational morphism from a smooth projective surface Y. Let $p : Y \to C$ be a projective surjective morphism onto a projective smooth curve C with the genus $g(C) \ge 1$. Then every f-exceptional curve E on Y is contained in a fiber of $p : Y \to C$.*

Sketch of the proof. By taking suitable blow-ups, we may assume that E is smooth. Let $\{E_i\}_{i \in I}$ be the set of all f-exceptional divisors. Suppose that $p(E) = C$. We consider the subgroup G of $\mathrm{Pic}(E)(k)$ generated by $\{\mathcal{O}_E(E_i)\}_{i \in I}$. Since $k \neq \overline{\mathbf{F}}_p$, $(\pi^* \mathrm{Pic}^0(C))(k) \otimes_{\mathbf{Z}} \mathbf{Q} \setminus G \otimes_{\mathbf{Z}} \mathbf{Q}$ is not empty where $\pi = p|_E : E \to C$. Here, we used the fact that the rank of $(\pi^* \mathrm{Pic}^0(C))(k)$ is infinite since $k \neq \overline{\mathbf{F}}_p$ (see [FJ, Theorem 10.1]). On the other hand, $(\pi^* \mathrm{Pic}^0(C))(k) \otimes_{\mathbf{Z}} \mathbf{Q} \subset G \otimes_{\mathbf{Z}} \mathbf{Q}$ since X is $\mathbf{Q}$-factorial. It is a contradiction. Therefore, E is in a fiber of $p : Y \to C$. □

Theorem 4.2 does not hold when $k = \overline{\mathbf{F}}_p$.

Example 4.3. We consider $C = (x^3 + y^3 + z^3 = 0) \subset \mathbf{P}^2 = H$, which is a hyperplane in $\mathbf{P}^3$, over $k = \overline{\mathbf{F}}_p$ with $p \neq 3$. Let X be the cone over C in $\mathbf{P}^3$ with the vertex P. Let $Z \to \mathbf{P}^3$ be the blow-up at P and let Y be the strict transform of X. Then Y is a $\mathbf{P}^1$-bundle over C, the singularity of X is not rational, X is $\mathbf{Q}$-factorial (see Theorem 4.1), and $f : Y \to X$ contracts a section of $p : Y \to C$.

If $k = \overline{\mathbf{F}}_p$, then we can easily obtain the finite generation of sectional rings.

Theorem 4.4. *Assume that $k = \overline{\mathbf{F}}_p$. Let X be a projective surface and let D be a Weil divisor on X. Then the sectional ring*

$$R(D) = \bigoplus_{m \ge 0} H^0(X, \mathcal{O}_X(mD))$$

is a finitely generated k-algebra.

Sketch of the proof. As in the proof of Corollary 1.3, we may assume that D is big. By contracting curves C with $D \cdot C < 0$, we may further assume that D is nef and big. Then D is semi-ample by Corollary 2.3. Therefore, $R(D)$ is finitely generated. □

The geometry over $\overline{\mathbf{F}}_p$ seems to be completely different from that over $k \neq \overline{\mathbf{F}}_p$. The minimal model theory for log surfaces over $k = \overline{\mathbf{F}}_p$ is discussed in [Ta1, Part 2], which has a slightly different flavor from that over $k \neq \overline{\mathbf{F}}_p$.

5. Indecomposable curves of canonical type. In this section, we discuss a key result for the proof of the abundance theorem for $\kappa = 0$: Theorem 5.1. Note that the abundance theorem for $\kappa = 1$ is easy to prove and the abundance theorem for $\kappa = 2$ has already been treated in Theorem 3.1.

Theorem 5.1 (cf. [Fn, Theorem 6.2] and [Ta1, Theorem 7.5])**.** *Let (X, Δ) be a $\mathbf{Q}$-factorial projective log surface such that Δ is a $\mathbf{Q}$-divisor. Assume that $K_X + \Delta$ is nef and $\kappa(X, K_X + \Delta) = 0$. Then $K_X + \Delta \sim_{\mathbf{Q}} 0$.*

Let us recall the definition of *indecomposable curves of canonical type* in the sense of Mumford.

Definition 5.2 (Indecomposable curves of canonical type)**.** Let X be a smooth projective surface and let Y be an effective divisor on X. Let $Y = \sum_{i=1}^{k} n_i Y_i$ be the prime decomposition. We say that Y is an *indecomposable curve of canonical type* if $K_X \cdot Y_i = Y \cdot Y_i = 0$ for every i, Supp$Y$ is connected, and the greatest common divisor of integers $n_1, \cdots, n_k$ is equal to one.

The following proposition is a key result. For a proof, see, for example, [M, Lemma] and the proof of [To, Theorem 2.1]. See also [Ta1, Proposition 7.3].

Proposition 5.3. *Let X be a smooth projective surface over k and let Y be an indecomposable curve of canonical type. If $\mathcal{O}_Y(Y)$ is torsion and $H^1(X, \mathcal{O}_X) = 0$, then Y is semi-ample and $\kappa(X, Y) = 1$. If $\mathcal{O}_Y(Y)$ is torsion and* $\mathrm{char}(k) > 0$, *then Y is always semi-ample and $\kappa(X, Y) = 1$ without assuming $H^1(X, \mathcal{O}_X) = 0$. Therefore, if $k = \overline{\mathbf{F}}_p$, then Y is semi-ample and $\kappa(X, Y) = 1$ since $\mathcal{O}_Y(Y)$ is always torsion.*

For the details of our proof of the abundance theorem for $\kappa = 0$, that is, Theorem 5.1, see [Ta1, Section 7].

6. Relative vanishing theorems. The following theorem is a special case of [KK, 2.2.5 Corollary] (see also [Ko, Theorem 9.4] and [Ta2, Sections 2 and 4]). Note that it holds over any algebraically closed field. We also note that the Kodaira vanishing theorem does not always hold for surfaces if the characteristic of the base field is positive.

Theorem 6.1 (Relative vanishing theorem)**.** *Let $\varphi : V \to W$ be a proper birational morphism from a smooth surface V to a normal surface W. Let $\mathcal{L}$ be a line bundle on V. Assume that*

$$\mathcal{L} \equiv_{\varphi} K_V + E + N$$

where $\equiv_{\varphi}$ denotes the relative numerical equivalence, E is an effective φ-exceptional $\mathbf{R}$-divisor on V such that $\llcorner E \lrcorner = 0$, and N is a φ-nef $\mathbf{R}$-divisor on V. Then $R^1\varphi_*\mathcal{L} = 0$.

As an application of Theorem 6.1, we obtain Theorem 6.2, whose formulation is suitable for our theory of log surfaces.

Theorem 6.2. *Let (X, Δ) be a log surface. Let $f : X \to Y$ be a proper birational morphism onto a normal surface Y. Assume that one of the following conditions holds.*
(1) *$-(K_X + \Delta)$ is f-ample.*
(2) *$-(K_X + \Delta)$ is f-nef and $\llcorner \Delta \lrcorner = 0$.*
Then $R^1 f_*\mathcal{O}_X = 0$.

Proof. Without loss of generality, we may assume that Y is affine. When $-(K_X + \Delta)$ is f-ample, by perturbing the coefficients of Δ, we may assume that $\llcorner \Delta \lrcorner = 0$. More precisely, let H be an f-ample Cartier divisor on X. Then we can find an effective $\mathbf{R}$-divisor Δ' on X such that $\llcorner \Delta' \lrcorner = 0$, $\Delta' \sim_{\mathbf{R}} \Delta + \varepsilon H$ for a sufficiently small positive real number ε, and $-(K_X + \Delta')$ is f-ample. Let $\varphi : Z \to X$ be the minimal resolution of X. We set $K_Z + \Delta_Z = \varphi^*(K_X + \Delta)$. Note that Δ_Z is effective. Then we have $-\llcorner \Delta_Z \lrcorner = K_Z + \{\Delta_Z\} - \varphi^*(K_X + \Delta)$. By Theorem 6.1, $R^1\varphi_*\mathcal{O}_Z(-\llcorner \Delta_Z \lrcorner) = R^1(f \circ \varphi)_*\mathcal{O}_Z(-\llcorner \Delta_Z \lrcorner) = 0$. We note that we can write $\{\Delta_Z\} = E + M$ where E is a φ-exceptional (resp. $(f \circ \varphi)$-exceptional) effective $\mathbf{R}$-divisor with $\llcorner E \lrcorner = 0$ and M is an effective $\mathbf{R}$-divisor such that every irreducible component of M is not φ-exceptional (resp. $(f \circ \varphi)$-exceptional). In this case, M is φ-nef (resp. $(f \circ \varphi)$-nef). Since $0 \to \mathcal{O}_Z(-\llcorner \Delta_Z \lrcorner) \to \mathcal{O}_Z \to \mathcal{O}_{\llcorner \Delta_Z \lrcorner} \to 0$, we obtain $0 \to \varphi_*\mathcal{O}_Z(-\llcorner \Delta_Z \lrcorner) \to \mathcal{O}_X \to \varphi_*\mathcal{O}_{\llcorner \Delta_Z \lrcorner} \to 0$. Since $\llcorner \Delta \lrcorner = 0$, $\llcorner \Delta_Z \lrcorner$ is φ-exceptional. Therefore, $\varphi_*\mathcal{O}_{\llcorner \Delta_Z \lrcorner}$ is a skyscraper sheaf on X. Thus, we obtain $\cdots \to R^1 f_*(\varphi_*\mathcal{O}_Z(-\llcorner \Delta_Z \lrcorner)) \to R^1 f_*\mathcal{O}_X \to 0$. Since $R^1 f_*(\varphi_*\mathcal{O}_Z(-\llcorner \Delta_Z \lrcorner)) \subset R^1(f \circ \varphi)_*\mathcal{O}_Z(-\llcorner \Delta_Z \lrcorner) = 0$, we obtain $R^1 f_*\mathcal{O}_X = 0$. □

We close this section with the following important results. For definitions, see [KM, Notation 4.1].

Proposition 6.3 (cf. [KM, Proposition 4.11] and [Fn, Proposition 3.5])**.** *Let X be an algebraic* *surface defined over an algebraically closed field k of arbitrary characteristic.*

(a) *Let (X, Δ) be a numerically dlt pair. Then every Weil divisor on X is* **Q**-*Cartier, that is, X is* **Q**-*factorial.*

(b) *Let (X, Δ) be a numerically lc pair. Then it is lc.*

The proof given in [Fn] works over any algebraically closed field once we adopt Artin's lemmas (see [B, Lemmas 3.3 and 3.4]) instead of [KM, Theorem 4.13] since the relative Kawamata–Viehweg vanishing theorem holds by Theorem 6.1.

Theorem 6.4 (cf. [KM, Theorem 4.12])**.** *Let X be an algebraic surface defined over an algebraically closed filed k of arbitrary characteristic. Assume that (X, Δ) is numerically dlt. Then X has only rational singularities.*

Theorem 6.4 follows from Theorem 6.2 (2).

Remark 6.5. The proof of Proposition 6.3 uses the classification of the dual graphs of the exceptional curves of log canonical surface singularities. In the framework of [Ta1], we do not need Proposition 6.3 or the classification of log canonical surface singularities even for the minimal model theory of log canonical surfaces (see [Ta1, Part 3]). So, we are released from the classification of log canonical surface singularities when we discuss the minimal model theory of log surfaces.

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