The Caffarelli-Kohn-Nirenberg type inequalities involving critical and supercritical weights

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Abstract: The main purpose of this article is to establish the CKN-type inequalities for all $\alpha \in \mathbf{R}$ and to study the relating matters systematically. Roughly speaking, we shall discuss about the characterizations of the CKN-type inequalities for all $\alpha \in \mathbf{R}$ as the variational problems, the existence and nonexistence of the extremal solutions to these variational problems in proper spaces, the exact values and the assymptotic behaviors of the best constants $S(p, q, \alpha)$ and $C(p, q)$.

Key words: CKN-type inequality; Hardy-Sobolev inequality; weighted Hardy inequality; degenerate elliptic equation; best constant.

1. Introduction. We shall establish the CKN-type inequalities for all $\alpha \in \mathbf{R}$ and study the relating matters systematically.

In the noncritical case ($\alpha \neq 1 - \frac{n}{p}$) the CKN-type inequalities with best constants $S(p, q, \alpha)$ are represented by

$$\text{(1.1)} \qquad \int_{\mathbf{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx \geq S(p, q, \alpha) \left(\int_{\mathbf{R}^n} |u|^q |x|^{\beta q}\, dx \right)^{p/q}$$

for any $u \in W^{1,p}_{\alpha,0}(\mathbf{R}^n)$ with $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n})$. Here $n \geq 1$, $1 \leq p < +\infty$ and q, α, β are real numbers satisfying the noncritical relation (NCR) given by Definition 2.6. On the other hand in the critical case ($\alpha = 1 - \frac{n}{p}$), the CKN-type inequalities with best constants $C(p, q)$ become

$$\text{(1.2)} \qquad \int_{B_1} |\nabla u|^p |x|^{p-n}\, dx \geq C(p, q) \left(\int_{B_1} \frac{|u|^q}{|x|^n A_1(|x|)^{q+1-\frac{q}{p}}}\, dx \right)^{p/q}$$

for any $u \in W^{1,p}_{\alpha,0}(B_1)$. Here B_1 is a unit ball having its center at the origin, $A_1(t) = \log \frac{R}{t}$ for $R > 1$ if $n \geq 2$, and $A_1(t) = \log \frac{1}{t}$ if $n = 1$, and the parameters should obey the critical relation (CR) given by Definition 2.7. Roughly speaking, we shall discuss about the characterizations of the CKN-type inequalities for all $\alpha \in \mathbf{R}$ as the variational problems, the existence and nonexistence of the extremal solutions to these variational problems in proper spaces, the exact values and the assymptotic behaviors of the best constants $S(p, q, \alpha)$ and $C(p, q)$, and so on. When $p = 2$ and $\alpha > 1 - \frac{n}{2}$, these topics were already treated by F. Catrina and Z. Wang in [6] and they obtained interesting results (See also [10], [7], [1] and [8]). In these problems, the presence of weight functions in the both sides prevents us from employing effectively the so-called spherically symmetric rearrangement, and the invariance of $\mathbf{R}^n$ by the group of dilatations creates some possible loss of compactness. Instead of the full proofs, the important remarks in the proofs are given just after each theorems. The full proofs will be given in the coming paper [11].

Before we go further into our main results on the CKN-type inequalities involving critical and supercritical cases, we give a brief historical review here. The inequality (1.1) for $\alpha > 1 - \frac{n}{p}$ is often called the Caffarelli-Kohn-Nirenberg type (the CKN-type inequalities). In fact in [5] they established general multiplicative inequalities including these types. In [9] we have also studied these inequalities among more general imbedding theorems on the weighted Sobolev spaces, where the weights are powers of distance from a given closed set F.

It was also very interesting for us to study further the properties of the imbedding operators

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obtained there. But for a general F it seemed not easy to study these problems in a detailed way. By this reason, in [10] we restricted ourselves on the simplest case that F consists of a single point, namely, the origin. In this particular case we have studied the relating problems in a various aspect and obtained interesting results such as the exact values of the best constant $S = S(p, q, \alpha)$ in certain cases, the existence and nonexistence of the extremals and so on.

Recently we have revisited the weighted Hardy-Sobolev inequality in [4] and [3]. It is easy to see that the classical CKN-type inequality coincides with the weighted Hardy-Sobolev inequality if $\beta = \alpha - 1$, or equivalently $p = q$. To our surprise it was shown that the weighted Hardy-Sobolev inequalities themselves hold for all $\alpha \in \mathbf{R}$ with some modifications. In fact, even if $\alpha = 1 - \frac{n}{p}$ holds, the sharp inequality of the Hardy type remains valid as long as the whole space $\mathbf{R}^n$ is replaced by a bounded domain containing the origin and the weight functions in the right hand side are replaced by the logarismic ones. Moreover we have successfully improved those weighted Hardy-Sobolev inequalities by finding out sharp missing terms, as a result they turned out to be very useful in many aspects. For the complete argument and the related applications see [4].

On the other hand, the counterpart in the CKN-type inequalities to the weighted Hardy-Sobolev inequalities in [4] seems to be unknown so far. But it seems reasonable for us to expect that the CKN-type inequalities should remain valid for all $\alpha \in \mathbf{R}$ with a similar modification as was performed in the weighted Hardy-Sobolev inequalities. In this spirit we shall establish the CKN type inequalities for all $\alpha \in \mathbf{R}$, and as the application we shall further study the relating topics to the CKN type inequalities for all $\alpha \in \mathbf{R}$ systematically in the present paper.

2. Function spaces and related properties.

Definition 2.1. Let Ω be a domain of $\mathbf{R}^n$. For a nonnegative measurable function ω on Ω, let $L^p(\Omega, \omega)$ denote the space of Lebesgue measurable functions defined on Ω, for which

$$(2.1) \quad \|u\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |u|^p \omega\, dx\right)^{1/p} < +\infty.$$

Definition 2.2. Let p and α satisfy $1 \le p < +\infty$ and $\alpha \neq 1 - \frac{n}{p}$. Let Ω be a domain of $\mathbf{R}^n$ such that $0 \in \Omega$. Then, by $W^{1,p}_{\alpha,0}(\Omega)$ we denote the completion of $C_0^\infty(\Omega \setminus \{0\})$ with respect to a norm defined by

$$(2.2) \quad \|u\|_{W^{1,p}_{\alpha,0}(\Omega)} = \|\,|\nabla u|\,\|_{L^p(\Omega,|x|^{p\alpha})} + \|u\|_{L^p(\Omega,|x|^{p(\alpha-1)})}.$$

Definition 2.3. Let p and α satisfy $1 \le p < +\infty$ and $\alpha = 1 - \frac{n}{p}$. Let Ω be a bounded domain of $\mathbf{R}^n$ such that $0 \in \Omega$ and let R be a positive number such that $R > \sup_{x\in\Omega} |x|$. Then, by $W^{1,p}_{\alpha,0}(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ with respect to a norm defined by

$$(2.3) \quad \|u\|_{W^{1,p}_{\alpha,0}(\Omega)} = \|\,|\nabla u|\,\|_{L^p(\Omega,|x|^{p\alpha})} + \|u\|_{L^p(\Omega,|x|^{-n}A_1(|x|)^{-p})}.$$

Here $A_1(t) = \log \frac{R}{t}$. When $n = 1$, we also treat the space $W^{1,p}_{\alpha,0}(\Omega)$ with $R \ge \sup_{x\in\Omega} |x|$.

Definition 2.4 (Definition of $\beta(p, q, \alpha)$)**.** For any $p, q \ge 1$, $\alpha \in \mathbf{R}$ and $n \ge 1$ we set

$$(2.4) \quad \beta(p, q, \alpha) = n\left(\frac{1}{p} - \frac{1}{q}\right) + \alpha - 1.$$

We shall classify the CKN-type inequalities according to the range of the parameter α into the three cases. Namely

Definition 2.5. The parameter α is said to be subcritical, critical and supercritical if α satisfies $\alpha > 1 - \frac{n}{p}$, $\alpha = 1 - \frac{n}{p}$ and $\alpha < 1 - \frac{n}{p}$ respectively.

In the next we define the noncritical (*i.e.* subcritical or supercritical) and the critical relations.

Definition 2.6 (The noncritical relation (NCR))**.** The parameters p, q, n, α and β are said to satisfy the noncritical relation (NCR) if they satisfy

$$(2.5) \quad \begin{cases} \alpha \neq 1 - \dfrac{n}{p}, \\ \beta = \beta(p, q, \alpha), \\ q < +\infty \\ \alpha - 1 \le \beta \le \alpha. \end{cases}$$

Definition 2.7 (The critical relation (CR))**.** The parameters p, q, n, α and β are said to satisfy the critical relation (CR) if they satisfy

$$(2.6) \quad \begin{cases} \alpha = 1 - \dfrac{n}{p}, \\ \beta = -\dfrac{n}{q} \quad \left(= \beta\left(p, q, 1 - \dfrac{n}{p}\right)\right), \\ q < +\infty \\ \alpha - 1 \le \beta \le \alpha \end{cases}$$

For given p and n, q is the only living parameter in the critical case. Further from the remaining conditions we see that

$$(2.7)\quad \begin{cases} -\dfrac{n}{p} \le \beta \le 1-\dfrac{n}{p}, \quad p \le q \le \dfrac{np}{n-p} \\ \qquad\qquad\qquad\qquad \text{if } 1 \le p < n, \\ -\dfrac{n}{p} \le \beta < 0, \quad p \le q < +\infty, \\ \qquad\qquad\qquad\qquad \text{if } p \ge n. \end{cases}$$

It is helpful for us to know in advance the existence of the continuous imbedding operators among our spaces. Namely from Theorem 4.1 and Theorem 5.1 we have the following proposition that is very fundamental in the present work.

Proposition 2.1. *Let p satisfy $1 \le p < +\infty$ and let n satisfy $n \ge 1$.*

1. *Assume that the parameters p, q, n, α and β satisfy the noncritical relation (NCR). Then we have*
$$W^{1,p}_{\alpha,0}(\mathbf{R}^n) \subset L^q(\mathbf{R}^n, |x|^{\beta q}).$$
In addition $C^\infty_0(\mathbf{R}^n \setminus \{0\})$ is densely contained in $W^{1,p}_{\alpha,0}(\mathbf{R}^n)$.
2. *Assume that $\alpha = 1 - \frac{n}{p}$ (the critical case). Then it holds that*
$$W^{1,p}_{\alpha,0}(B_1) \subset L^p(B_1, |x|^{-n} A_1(|x|)^{-p}),$$
where B_1 is a unit ball with a center being the origin. $A_1(t) = \log \frac{R}{t}$ for $R > 1$ if $n \ge 2$, and $A_1(t) = \log \frac{1}{t}$ if $n = 1$.
In addition, both $C^\infty_0(B_1)$ and $C^\infty_0(B_1 \setminus \{0\})$ are densely contained in $W^{1,p}_{\alpha,0}(B_1)$.

Lastly we define a Banach space of radial functions.

Definition 2.8. For a unit ball $B_1 = \{x \in \mathbf{R}^n : |x| < 1\}$, we set

$$(2.8)\quad \begin{cases} W^{1,p}_{\alpha,0}(B_1)_{\mathrm{rad}} \\ \quad = \{u \in W^{1,p}_{\alpha,0}(B_1) : u \text{ is a radial function}\}, \\ \|u\|_{W^{1,p}_{\alpha,0}(B_1)_{\mathrm{rad}}} = \|u\|_{W^{1,p}_{\alpha,0}(B_1)}. \end{cases}$$

3. Variational problems and some notations.

Definition 3.1. For the noncritical case ($\alpha \ne 1 - \frac{n}{p}$) we set for $\beta = \beta(p,q,\alpha)$

$$(3.1)\quad E^{p,q,\alpha}(u) = \frac{\int_{\mathbf{R}^n} |\nabla u|^p |x|^{p\alpha}\, dx}{\left(\int_{\mathbf{R}^n} |u|^q |x|^{\beta q}\, dx\right)^{p/q}}.$$
$$(u \in W^{1,p}_{\alpha,0}(\mathbf{R}^n) \setminus \{0\})$$

For the critical case ($\alpha = 1 - \frac{n}{p}$) we set

$$(3.2)\quad F^{p,q}(u) = \frac{\int_{B_1} |\nabla u|^p |x|^{p-n}\, dx}{\left(\int_{B_1} \frac{|u|^q |x|^{-n}}{A_1(|x|)^{q+1-\frac{q}{p}}}\, dx\right)^{p/q}}.$$
$$(u \in W^{1,p}_{\alpha,0}(B_1) \setminus \{0\})$$

Here $A_1(t) = \log \frac{R}{t}$ for $R > 1$ if $n \ge 2$, and $A_1(t) = \log \frac{1}{t}$ if $n = 1$.

Definition 3.2. Under the condition (NCR) we set

$$S(p,q,\alpha) = \inf_{u \in W^{1,p}_{\alpha,0}(\mathbf{R}^n) \setminus \{0\}} E^{p,q,\alpha}(u) \qquad (P)$$

$$S_{\mathrm{rad}}(p,q,\alpha) = \inf_{u \in W^{1,p}_{\alpha,0}(\mathbf{R}^n)_{\mathrm{rad}} \setminus \{0\}} E^{p,q,\alpha}(u) \qquad (P_{\mathrm{rad}})$$

Secondly we consider the critical case.

Definition 3.3. Under the condition (CR) we set

$$C(p,q) = \inf_{u \in W^{1,p}_{\alpha,0}(B_1) \setminus \{0\}} F^{p,q}(u). \qquad (P^c)$$

$$C_{\mathrm{rad}}(p,q) = \inf_{u \in W^{1,p}_{\alpha,0}(B_1)_{\mathrm{rad}} \setminus \{0\}} F^{p,q}(u). \qquad (P^c_{\mathrm{rad}})$$

We prepare the following notations.

Definition 3.4. For (α, β) with $\beta = \beta(p,q,\alpha)$, let us set

$$(3.3)\quad \begin{cases} \overline{\alpha} = 2 - \alpha - \dfrac{2n}{p}, \\ \underline{\beta} = -\beta - \dfrac{2n}{q}. \end{cases}$$

Under these notation we see that

$$(3.4)\quad \underline{\beta} = \beta(p,q,\overline{\alpha}).$$

Further we see immediately that $(\overline{\alpha}, \underline{\beta})$ is symmetric to (α, β) with respect to a point $(1 - \frac{n}{p}, -\frac{n}{q})$ in $\mathbf{R}^2$, and that $1 - \alpha + \beta = 1 - \overline{\alpha} + \underline{\beta}$ is satisfied.

Definition 3.5. Let p satisfy $1 \le p$. For any subset $A \subset \mathbf{R}$ and $Q \subset [p, +\infty)$, considering the subset of $\mathbf{R}^2$

$$G = \{(\alpha,\beta) : \beta = \beta(p,q,\alpha), \alpha \in A, q \in Q\},$$

we set

$$(3.5)\quad G^{\diamondsuit} = \{(\overline{\alpha}, \underline{\beta}) : \underline{\beta} = \beta(p,q,\overline{\alpha}), \alpha \in A, q \in Q\},$$

where $(\overline{\alpha}, \underline{\beta})$ is given by (3.3).

4. Main results in the noncritical case.

In this subsection we consider the noncritical case ($\alpha \ne 1 - \frac{n}{p}$). For the sake of self-containedness, we shall state our results combining with some relating known results.

Theorem 4.1 (The imbedding results). *Let p satisfy $1 < p < +\infty$ and let n satisfy $n \geq 1$. Assume that the parameters p, q, n, α and β satisfy the non-critical relation (NCR). Then we have $S(p,q,\alpha) > 0$, namely it holds that for any $u \in W^{1,p}_{\alpha,0}(\mathbf{R}^n)$,*

$$\int_{\mathbf{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx \geq S(p,q,\alpha)\left(\int_{\mathbf{R}^n} |u|^q |x|^{\beta p}\, dx\right)^{p/q}. \tag{4.1}$$

Proof.
1. In the subcritical case the imbeding inequalities are known as the classical CKN-type inequalities. (See also the author's paper [10].)
2. In the supercritical case we employ the isometric transformation T from the spaces $W^{1,p}_{\alpha,0}(\mathbf{R}^n)$ and $L^q(\mathbf{R}^n, |x|^{\beta q}))$ to the spaces $W^{1,p}_{\overline{\alpha},0}(\mathbf{R}^n)$ and $L^q(\mathbf{R}^n, |x|^{\underline{\beta} q}))$ respectively with $\overline{\alpha} = 2 - \alpha - \frac{2n}{p}$ and $\underline{\beta} = -\beta - \frac{2n}{q}$.
3. When $p = 1$ and $\beta > \alpha - 1$, these inequalities remain valid, which are often called "the weighted isoperimetric inequality". □

Theorem 4.2 (The imbedding results in the radial function space). *Let p satisfy $1 < p < +\infty$ and let n satisfy $n \geq 1$. Assume that the parameters p, q, α, β and n satisfy the noncritical relation (NCR) and $\alpha - 1 < \beta \leq \alpha$. Then the best constant $S_{\mathrm{rad}}(p,q,\alpha)$ in the radial function space is achieved.*

Proof. When $\alpha = \beta = 0$, this was initially shown in [15]. In the subcritical case, this was already established in [10], and in the supercritical case we employ again the isometric transformation T. □

Theorem 4.3 (The continuity of the best constant). *Let p satisfy $1 < p < +\infty$ and let n satisfy $n \geq 1$. Assume that the parameters p, q, n, α and β satisfy the noncritical relation (NCR). Then we have the followings:*
1. *It holds that*
$$S(p,q,\alpha) = S(p,q,\overline{\alpha}). \tag{4.2}$$
2. *$S(p,q,\alpha)$ is continuous on q, α. In particular we have*
$$\lim_{q \to p+0} S(p,q,\alpha) = \Lambda_{p,\alpha}. \tag{4.3}$$
Here $\Lambda_{p,\alpha} = |\frac{n-p+\alpha p}{p}|^p$ is the best constant for the weighted Hardy-Sobolev inequality.

In the next we see that the best constant $S(p,q,\alpha)$ is attained by some elements in $W^{1,p}_{\alpha,0}(\mathbf{R}^n)$ provided that $\alpha - 1 < \beta = \beta(p,q,\alpha) < \alpha$ is satisfied.

Theorem 4.4 (Existence and Nonexistence of extremals). *Let p satisfy $1 < p < +\infty$ and let n satisfy $n \geq 1$. Assume that the parameters p, q, n, α and β satisfy the noncritical relation (NCR). Then we have the followings:*
1. *Assume that $\alpha - 1 < \beta < \alpha$. Then the best constant $S(p,q,\alpha)$ is achieved by some $u \in W^{1,p}_{\alpha,0}(\mathbf{R}^n)$.*
2. *For $\beta = \alpha - 1$ or equivalently $p = q$, $S(p,p,\alpha) = \Lambda_{p,\alpha}$ holds, and $S(p,p,\alpha)$ is not achieved.*
3. *Assume that $p = 2$ and $n > 2$. If either $\alpha > 0$ or $\alpha < 2(1 - \frac{n}{p})$, then $S(2,2^*,\alpha) = S(2,2^*,0)$ holds and $S(2,2^*,\alpha)$ is not achieved. Here $2^* = \frac{2n}{n-2}$, $\beta(2,2^*,\alpha) = \alpha$ and $\beta(2,2^*,0) = 0$.*

Proof.
1. When $p = 2$ and $\alpha > 1 - \frac{n}{2}$, these topics were already treated by F. Catrina and Z. Wang in [6]. They studied the CKN-type inequality with $p = 2$ and $\alpha > 1 - \frac{n}{2}$ intensively and obtained interesting results (See also [10], [7] and [8]).
2. In the study of the existence and nonexistence of extremals of the CKN-type inequalities, as in the study of the Sobolev inequality, the most of difficulty come from the lack of the compactness of the imbedding operators. However, our aim is accomplished by using the multiplicative inequality, the sophisticated compactness lemma, the sharp Fatou's lemma and a modified concentration-compactness lemma essentially due to P.L. Lions [14] (see also [12,13]). When $p = 2$, in the present paper the proof becomes rather simple without the concentration-compactness argument.
3. In the assertion 3, it follows from Theorem 4.2 and Theorem 4.6 that $S(2,2^*,\alpha)$ is achieved if and only if $2(1 - \frac{n}{p}) \leq \alpha \leq 0$ with $\alpha \neq 1 - \frac{n}{p}$. □

Theorem 4.5 (The asymptotic behavior of the best constant). *Let p satisfy $1 < p < n$ and let n satisfy $n \geq 1$. Assume that the parameters p, q, n, α and β satisfy the noncritical relation (NCR) and assume that either $\alpha \geq 0$ or $\alpha \leq 2(1 - \frac{n}{p})$. Then we have the followings:*
1. *For $\alpha = \beta$, we have*
$$l(p,\alpha,n) S(p,p^*,0) \leq S(p,p^*,\alpha) \leq S(p,p^*,0). \tag{4.4}$$
Here $p^* = \frac{np}{n-p}$ and

$$(4.5)\qquad l(p,\alpha,n)=\left(\frac{|n-p+\alpha p|}{|n-p+\alpha p|+|\alpha|p}\right)^p>0$$

2. *There is a positive number $m(p,\alpha,n)$ such that we have for any α and β satifying $|\alpha|\geq m(p,\alpha,n)$ and $\alpha-1<\beta=\beta(p,q,\alpha)\leq\alpha$,*

$$(4.6)\qquad S(p,q,\alpha)\geq\left(\Lambda_{p,\alpha}\right)^{p(\alpha-\beta)}S(p,p^*,\alpha)^{1-\alpha+\beta}$$

Proof.

1. By $S(p,p^*,0)$ we denote the usual Sobolev best constant without weights. Here note that $\beta(p,p^*,0)=0$.
2. We note that $\lim_{|\alpha|\to\infty}l(p,\alpha,n)=\frac{1}{2^p}$. □

Theorem 4.6 (The symmetricity of the extremals)**.** *Let p satisfy $1<p<+\infty$ and let n satisfy $n\geq 1$.*

1. *Assume that $p<n$ and $n\geq 2$. Let A be the subset of $\mathbf{R}^2$ given by*

$$\Big\{(\alpha,\beta)\in\mathbf{R}^2;\beta=\beta(p,q,\alpha),1-\frac{n}{p}<\alpha\leq 0,\ p\leq q\leq p^*\Big\},$$

where $p^*=\frac{np}{n-p}$. Then we have for any (p,q,α,β,n) satisfying (NCR) with $(\alpha,\beta)\in A\cup A^{\diamondsuit}$, $S(p,q,\alpha)=S_{\mathrm{rad}}(p,q,\alpha)$.

2. *Assume that $n\geq 2$. Let B be the subset of $\mathbf{R}^2$ given by*

$$\Big\{(\alpha,\beta)\in\mathbf{R}^2;\beta=\beta(p,q,\alpha),\max\Big(0,1-\frac{n}{p}\Big)<\alpha\leq 1-\frac{1}{p},p\leq q\leq\frac{p}{p-1}\cdot\frac{np-n-p\alpha}{n-p+p\alpha}\Big\}$$

Then we have for any (p,q,α,β,n) satisfying (NCR) with $(\alpha,\beta)\in B\cup B^{\diamondsuit}$, $S(p,q,\alpha)=S_{\mathrm{rad}}(p,q,\alpha)$.

Proof.

1. In the assertion 2, the condition $\alpha\leq 1-\frac{1}{p}$ is needed to have a nonempty set B. Moreover the condition on q is equivalent to the condition

$$\alpha-1\leq\beta(p,q,\alpha)\leq\frac{\alpha(n-p+\alpha p)}{n+\alpha p-np}.$$

2. The proof of this assertion will be established by the aid of the rearrangement argument involving weight functions. □

5. Main results in the critical case. In the critical case, no imbedding inequality of Sobolev type holds in the whole domain $\mathbf{R}^n$. But we will have the following imbedding inequalities in a ball B_1.

Theorem 5.1 (The imbedding results)**.** *Let p satisfy $1<p<+\infty$. Assume the critical relation (CR). Then we have the followings.*

1. *If $n\geq 2$, then there exist $R_0>1$ and a positive number $C(p,q)$ such that we have for any $u\in W^{1,p}_{\alpha,0}(B_1)$,*

$$(5.1)\qquad \int_{B_1}|\nabla u|^p|x|^{p-n}\,dx\geq C(p,q)\left(\int_{B_1}\frac{|u|^q}{|x|^n A_1(|x|)^{q+1-\frac{q}{p}}}\,dx\right)^{p/q}.$$

Here $A_1(t)=\log\frac{R}{t}$ for $R\geq R_0>1$ if $n\geq 2$. Moreover the weight function of the term in the right-hand side is sharp.

2. *If $n=1$, there exists a positive number $C(p,q)$ such that we have for any $u\in W^{1,p}_{\alpha,0}(B_1)$,*

$$(5.2)\qquad \int_{B_1}|\nabla u|^p|x|^{p-n}\,dx\geq C(p,q)\left(\int_{B_1}\frac{|u|^q}{|x|^n(\log\frac{1}{|x|})^{q+1-\frac{q}{p}}}\,dx\right)^{p/q}.$$

Moreover the weight function of the term in the right-hand side is sharp.

Proof.

1. When $n\geq 2$ and $p\neq q$, we can not replace $R>1$ by 1.
2. When $p\geq n$, we shall essentially exploit the decreasing rearrangement method with respect to a positive measure. By the techinical reason we need to assume $R\geq R_0$ for some $R_0>1$ provided that $n>1$. On the other hand, we shall employ the nonlinear potential theory in [2] when $1<p<n$, and R can be any number >1 in this case.
3. If B_1 is replaced by the whole domain $\mathbf{R}^n$, then the inequality does not hold. This fact is seen by the capacitary argument.
4. If the unit ball B_1 is replaced by any bounded domain Ω containing the origin, then the inequality remains valid with some $R>1$, and the proof is done in a similar way. □

Theorem 5.2 (The imbedding results in the radial function space)**.** *Let p satisfy $1<p<+\infty$ and let n satisfy $n\geq 1$. Assume the critical relation (CR). Then $C_{\mathrm{rad}}(p,q)$ is determined. Moreover $C_{\mathrm{rad}}(p,q)$ is achieved only if $n=1$.*

Proof. The exact value of $C_{rad.}(p,q)$ is known. □

Theorem 5.3 (The continuity of the best constant)**.** *Let p satisfy $1 \leq p < +\infty$. Assume the critical relation (CR). When $p > 1$, assume that $R \geq R_0 > 1$ if $n \geq 2$ and $R = 1$ if $n = 1$, where R_0 is the same constant given in Theorem* 5.1.

1. *For $p > 1$, $C(p,q)$ is positive and continuous on q for $-\frac{n}{q} \in [-\frac{n}{p}, 1-\frac{n}{p}]$ if $1 < p < n$, and on q for $-\frac{n}{q} \in [-\frac{n}{p}, 0)$ if $p \geq n$. In particular we have*

$$\lim_{q \to p+0} C(p,q) = \left(\frac{p-1}{p}\right)^p. \tag{5.3}$$

2. *If $p = 1$, then $C(p,q) = 0$ for any $R \geq 1$.*

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