A note on non-Robba p-adic differential equations

By Said MANJRA

Department of Mathematics, College of Science, Imam University,
P. O. Box: 240337, Riyadh 11322, Saudi Arabia

(Communicated by Masaki KASHIWARA, M.J.A., Feb. 14, 2011)

Abstract: Let $\mathcal{M}$ be a differential module, whose coefficients are analytic elements on an open annulus I ($\subset \mathbf{R}_{>0}$) in a valued field, complete and algebraically closed of inequal characteristic, and let $R(\mathcal{M}, r)$ be the radius of convergence of its solutions in the neighborhood of the generic point t_r of absolute value r, with $r \in I$. Assume that $R(\mathcal{M}, r) < r$ on I and, in the logarithmic coordinates, the function $r \longrightarrow R(\mathcal{M}, r)$ has only one slope on I. In this paper, we prove that for any $r \in I$, all the solutions of $\mathcal{M}$ in the neighborhood of t_r are analytic and bounded in the disk $D(t_r, R(\mathcal{M}, r)^-)$.

Key words: p-adic differential equations; Frobenius antecedent theorem.

1. Notations and Preliminaries. Let p be a prime number, $\mathbf{Q}_p$ the completion of the field of rational numbers for the p-adic absolute value $|.|$, $\mathbf{C}_p$ the completion of the algebraic closure of $\mathbf{Q}_p$, and Ω_p a p-adic complete and algebraically closed field containing $\mathbf{C}_p$ such that its value group is $\mathbf{R}_{\geq 0}$ and the residue class field is strictly transcendental over $\mathbf{F}_{p^\infty}$. For any positive real r, t_r will denote a generic point of Ω such that $|t_r| = r$. Let I be a bounded interval in $\mathbf{R}_{>0}$. We denote by $\mathcal{A}(I)$ the ring of analytic functions, on the annuli $\mathcal{C}(I) := \{a \in \Omega_p \mid |a| \in I\}$, $\mathcal{A}(I) = \Big\{\sum_{n\in\mathbf{Z}} a_n x^n \in \mathbf{C}_p[[x, 1/x]] \;\Big|\; \lim_{n\to\mp\infty} |a_n| r^n = 0, \forall\, r \in I\Big\}$, and by $\mathcal{H}(I)$ the completion of the ring of rational fractions f of $\mathbf{C}_p(x)$ having no pole in $\mathcal{C}(I)$ with respect to the norm $\|f\|_I := \sup_{r\in I} |f(t_r)|$. It is well known that $\mathcal{H}(I) \subseteq \mathcal{A}(I)$, with equality if I is closed. We define, for any $r \in I$, the absolute value $|.|_r$ over $\mathcal{A}(I)$ by $\Big|\sum_{n\in\mathbf{Z}} a_n x^n\Big|_r = \sup_{n\in\mathbf{Z}} |a_n| r^n$.

Let $R(I)$ denotes $\mathcal{A}(I)$ or $\mathcal{H}(I)$. A free $R(I)$-module $\mathcal{M}$ of finite rank μ is said to be $R(I)$-differential module if it is equipped with a $R(I)$-linear map $D : \mathcal{M} \to \mathcal{M}$ such that $D(am) = \partial(a)m + aD(m)$ for any $a \in R(I)$ and any $m \in \mathcal{M}$ where $\partial = d/dx$. To each $R(I)$-basis $\{e_i\}_{1\leq i\leq\mu}$ of $\mathcal{M}$ over $R(I)$ corresponds a matrix $G = (G_{ij}) \in M_\mu(R(I))$ satisfying $D(e_i) = \sum_{j=1}^{\mu} G_{ij} e_j$, called the matrix of ∂ with respect to the $R(I)$-basis $\{e_i\}_{1\leq i\leq\mu}$ or simply an associated matrix to $\mathcal{M}$, together with a differential system $\partial X = GX$ where X denotes a column vector $\mu \times 1$ or $\mu \times \mu$ matrix (see [2,3]). If $G' \in M_\mu(R(I))$ is the matrix of ∂ with respect to another $R(I)$-basis $\{e'_i\}_{1\leq i\leq\mu}$ of $\mathcal{M}$ and if $H = (H_{ij}) \in \mathrm{GL}_\mu(R(I))$ is the change of basis matrix defined by $e'_i = \sum_{i=1}^{\mu} H_{ij} e_i$ for all $1 \leq i \leq \mu$, it is known that:

- the matrices G and G' are related by the formula $G' = HGH^{-1} + \partial(H)H^{-1}$. The matrix $HGH^{-1} + \partial(H)H^{-1}$ is denoted $H[G]$.
- if Y is a solution matrix for the system $\frac{d}{dx}X = GX$ with coefficients in a differential field extension of $R(I)$, then the matrix HY is a solution matrix for $\frac{d}{dx}X = H[G]X$.

Generic radius of convergence. Let $\mathcal{M}$ be an $R(I)$-differential module of rank μ, $G = (G_{ij}) \in \mathrm{M}_\mu(R(I))$ an associated matrix to $\mathcal{M}$, $(G_n)_n$ a sequence of matrices defined by

$$G_0 = \mathbf{I}_\mu \quad \text{and} \quad G_{n+1} = \partial(G_n) + G_n G,$$

and $\|G\|_r = \max_{ij} |G_{ij}|_r$ be the norm of G associated to the absolute value $|.|_r$. For any $r \in I$, the quantity $R(\mathcal{M}, r) = \min(r, \liminf_{n\to\infty} \|G_n\|_r^{-1/n})$ represents the radius of convergence in the generic disc $D(t_r, r^-)$ of the solution matrix

2010 Mathematics Subject Classification. Primary 12H25.

doi: 10.3792/pjaa.87.40

$$\mathcal{U}_{G,t_r}(x) = \sum_{n\geq 0} \frac{G_n(t_r)}{n!}(x-t_r)^n$$

of the system $\frac{d}{dx}X = GX$ with $X(t_r) = \mathbf{I}_\mu$. We know that the function $r \to R(\mathcal{M}, r)$ is independent of the choice of basis and the ring $R(I)$ [3, Proposition 1.3], and the graph of the map $\rho \mapsto \log \circ R(\mathcal{M}, \exp(\rho))$, on any closed subinterval of I, is a concave polygon with rational slopes [5, Theorem 2]. This graph is called the generic polygon of the convergence of $\mathcal{M}$. The system $\partial X = GX$ is said to have an analytic and bounded solution in the disk $D(t_r, R(\mathcal{M}, r)^-)$ if

$$\sup_{n\geq 0}\left\|\frac{G_n}{n!}\right\|_r R(\mathcal{M}, r)^n < \infty.$$

The $R(I)$-differential module $\mathcal{M}$ is said to be non-Robba if $R(\mathcal{M}, r) < r$ for all $r \in I$.

Frobenius. Let $\varphi : \mathcal{C}(I) \to \mathcal{C}(I^p)$ be the Frobenius ramification $x \mapsto x^p$, where I^p is the image of I by the map $x \mapsto x^p$. A $R(I^p)$-differential module $\mathcal{N}$ is said to be a Frobenius antecedent of an $R(I)$-differential module $\mathcal{M}$ if $\mathcal{M}$ is isomorphic to the inverse image $\varphi^*\mathcal{N}$ of $\mathcal{N}$. In other words, if there exists a matrix $F \in \mathrm{M}_\mu(R(I^p))$ of the derivation d/dz (where $z = x^p$) in some $R(I^p)$-basis of $\mathcal{N}$ such that $px^{p-1}F(x^p)$ is a matrix of d/dx in some $R(I)$-basis of $\mathcal{M}$. The existence of such a Frobenius antecedent depends of the values of the function $r \mapsto R(\mathcal{M}, r)$. Recall the Frobenius structure theorem of Christol-Mebkhout [4, Theorem 4.1-4] where $\pi = p^{-1/p-1}$:

Theorem 1.1. *Let h be a positive integer and let $\mathcal{M}$ be a $R(I)$-differential module such that $R(\mathcal{M}, r) > r\pi^{1/p^{h-1}}$ for all $r \in I$. Then, there exists an $R(I^{p^h})$-differential module $\mathcal{N}_h$ such that $(\varphi^h)^*\mathcal{N}_h \cong \mathcal{M}$ and $R(\mathcal{M}, r)^{p^h} = R(\mathcal{N}_h, r^{p^h})$ for any $r \in I$, and $\mathcal{N}_h$ is called a Frobenius antecedent of order h of $\mathcal{M}$.*

In particular, if a $R(I)$-differential module $\mathcal{M}$ satisfies $R(\mathcal{M}, r) > r\pi$ for all $r \in I$, it has a Frobenius antecedent.

2. Main theorem. In this section, I denotes an open interval in $\mathbf{R}_{>0}$ and $\mathcal{M}$ a non-Robba $\mathcal{A}(I)$-differential module associated to some matrix $G \in \mathrm{M}_\mu(\mathcal{A}(I))$.

Theorem 2.1. *Assume that the generic polygon of convergence of $\mathcal{M}$ has only one slope. Then*

$$\sup_{n\geq 0}\left\|\frac{G_n}{n!}\right\|_r R(\mathcal{M}, r)^n < \infty \quad \textit{for all} \quad r \in I.$$

The proof of this theorem is based on the following lemmas:

Lemma 2.2. *Assume $R(\mathcal{M}, r) > \pi r$ for all $r \in I$ and let $\mathcal{N}$ be a Frobenius antecedent of $\mathcal{M}$. Let F be an associated matrix to $\mathcal{N}$ and assume there exists a real $r_0 \in I$ such that $\sup_{n\geq 0}\left\|\frac{F_n}{n!}\right\|_{r_0^p} R(\mathcal{N}, r_0^p)^n < \infty$. Then $\sup_{n\geq 0}\left\|\frac{G_n}{n!}\right\|_{r_0} R(\mathcal{M}, r_0)^n < \infty$.*

Proof. The matrix $\mathcal{V}(z) = (\mathcal{V}_{ij}(z))_{ij} = \mathcal{V}_{F,t^p_{r_0}}(z) = \sum_{n\geq 0} \frac{F_n(t^p_{r_0})}{n!}(z - t^p_{r_0})^n$ is the solution matrix of the differential system $\frac{d}{dz}V(z) = F(z)V(z)$ in the neighborhood of $t^p_{r_0}$ with $z = x^p$ and $\mathcal{V}(t^p_{r_0}) = \mathbf{I}_\mu$. The change of variables leads to $\frac{d}{dx}\mathcal{V}(x^p) = px^{p-1}F(x^p)\mathcal{V}(x^p)$. In addition, since $R(\mathcal{M}, r_0) > \pi r_0$, the map $x \mapsto x^p$ sends the closed disk $D(t_{r_0}, R(\mathcal{M}, r_0))$ into $D(t^p_{r_0}, R(\mathcal{M}, r_0)^p) = D(t^p_{r_0}, R(\mathcal{N}, r^p_0))$ [1, Lemma 3.1], and $\sup_{n\geq 0}\left\|\frac{F_n(t^p_{r_0})}{n!}\right\|.|x^p - t^p_{r_0}|^n = \sup_{n\geq 0}\left(\left\|\frac{F_n(t^p_{r_0})}{n!}\right\|.|x^{p-1} + x^{p-1}t_{r_0} + \ldots + t^{p-1}_{r_0}|^n.|x - t_{r_0}|^n\right) < \infty$ for all $x \in D(t_{r_0}, R(\mathcal{M}, r_0))$. In the neighborhood of t_{r_0}, the matrix $\mathcal{V}_{F,t^p_{r_0}}(x^p)$ can be written as $\mathcal{V}(x^p) = \sum_{n\geq 0} B_n(x - t_{r_0})^n$ where $B_n = (B_n(i,j))_{ij}$ are $\nu \times \nu$ matrices with entries un Ω. In that case, we have $\lim_{n\to\infty} |B_n(i,j)|\rho^n = 0$ for any $\rho < R(\mathcal{M}, r_0)$, and therefore

$$\begin{aligned}
\sup_{n\geq 0} |B_n(i,j)|\rho^n &= \sup_{x\in D(t_{r_0},\rho)} |\mathcal{V}_{ij}(x^p)| \\
&\leq \sup_{z\in D(t^p_{r_0},\rho^p)} |\mathcal{V}_{ij}(z)| \\
&= \sup_{n\geq 0}\left\|\frac{F_n(t^p_{r_0})}{n!}\right\|\rho^{pn}.
\end{aligned} \tag{2.1}$$

Since the matrix $\mathcal{V}(z)$ is analytic and bounded in $D(t^p_{r_0}, R(\mathcal{N}, r^p_0)^-)$, there exists a positive constant $C > 0$, by [2, Proposition 2.3.3], such that

$$\sup_{n\geq 0}\left\|\frac{F_n(t^p_{r_0})}{n!}\right\|\rho^{pn} < C \tag{2.2}$$

for any $\rho < R(\mathcal{M}, r_0)$ and close to $R(\mathcal{M}, r_0)$. Combining (2.1) and (2.2), and using again [2, Proposition 2.3.3], we find $\sup_{n\geq 0} |B_n(i,j)|R(\mathcal{M}, r_0)^n < \infty$ for all $1 \leq i, j \leq \nu$, and therefore, the matrix $\mathcal{V}(x^p)$ is analytic and bounded in the disk $D(t_{r_0}, R(\mathcal{M}, r_0)^-)$. In addition, since the matrix $px^{p-1}F(x^p)$ is associated to $\mathcal{M}$, then there exists an invertible matrix $H \in \mathrm{GL}_\mu(\mathcal{A}(I))$ (hence H is analytic and bounded in the disk $D(t_{r_0}, R(\mathcal{M}, r_0)^-)$) such that

$$G = H[px^{p-1}F(x^p)].$$

Thus, by [2, Proposition 2.3.2], the matrix $H\mathcal{V}(x^p)$ is a solution to the system $\partial X = GX$ in the neighborhood of t_{r_0}, and moreover it is analytic and bounded in the disk $D(t_{r_0}, R(\mathcal{M}, r_0)^-)$. This means that $\mathcal{U}_{G,t_{r_0}}(x) = H\mathcal{V}(x^p)H(t_{r_0})^{-1}$ is also analytic and bounded in the disk $D(t_{r_0}, R(\mathcal{M}, r_0)^-)$. □

Lemma 2.3. *The set of reals r in I for which*

$$\sup_{n\geq 0}\left\|\frac{G_n}{n!}\right\|_r R(\mathcal{M}, r)^n < \infty$$

is dense in I.

Proof. Let J be a closed subinterval of I not reduced to a point and let ρ be a real number in the interior of J. Then, by hypothesis, $R(\mathcal{M}, \rho)/\rho < 1$ and therefore there exists an integer h such that $\pi^{1/p^{h-1}} < R(\mathcal{M}, \rho)/\rho < \pi^{1/p^h}$. Since the function $r \mapsto R(\mathcal{M}, r)$ is continuous on J, there exists an open subinterval $J' \subset J$ containing ρ such that $\pi^{1/p^{h-1}} r < R(\mathcal{M}, r) < \pi^{1/p^h} r$ for all $r \in J'$.

There are two cases to consider:

Case 1: $h \leq 0$.

Let $\dot{\mathcal{H}}(J')$ be the quotient field of $\mathcal{H}(J')$. By cyclic vector lemma, we can associate $\dot{\mathcal{H}}(J') \otimes \mathcal{M}$ to a differential equation $\Delta(\dot{\mathcal{H}}(J) \otimes \mathcal{M}) = \partial^\mu + q_1(x)\partial^{\mu-1} + \cdots + q_\mu(x)$, where $q_i \in \dot{\mathcal{H}}(J')$ for $i = 1, \ldots, \mu$. Now pick a nonempty subinterval J'' of J' such that $q_i \in \mathcal{H}(J'')$ for $i = 1, \ldots, \mu$, and let r_0 be a real number in the interval J'' and $\lambda(r_0)$ be the maximum of the p-adic absolute values of the roots of the polynomial $\Delta(\dot{\mathcal{H}}(J) \otimes \mathcal{M}) = \lambda^\mu + q_1(t_{r_0})\lambda^{\mu-1} + \cdots + q_\mu(t_{r_0})$. Since $R(\mathcal{M}, r_0) = R(\dot{\mathcal{H}}(J) \otimes \mathcal{M}, r_0) < \pi^{1/p^h} r_0 < \pi r_0$, by virtue of [6, Theorem 3.1], we have $\log(R(\mathcal{M}, r_0)) = \frac{1}{p-1} + \log(\lambda(r_0))$ and all the solutions $u_1, \ldots, u_\mu$ of $\Delta(\dot{\mathcal{H}}(J) \otimes \mathcal{M})$ in the neighborhood of t_{r_0} are analytic and bounded in the disk $D(t_{r_0}, R(\mathcal{M}, r_0)^-)$. Now let W be the wronskian matrix of $(u_1, \ldots, u_\mu)$. Then, W is a solution of the system $\partial X = A_{\Delta(\dot{\mathcal{H}}(J)\otimes\mathcal{M})}X$ where

$$A_{\Delta(\dot{\mathcal{H}}(J)\otimes\mathcal{M})} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -q_\mu & -q_{\mu-1} & -q_{\mu-2} & \ldots & -q_1 \end{bmatrix}.$$

Moreover, by [2, Proposition 2.3.2], the matrix W is analytic and bounded in the disk $D(t_{r_0}, R(\mathcal{M}, r_0)^-)$. Since G and $A_{\Delta(\dot{\mathcal{H}}(J)\otimes\mathcal{M})}$ are associated to $\mathcal{H}(J'') \otimes \mathcal{M}$, there exists a matrix $H \in \mathrm{GL}_\mu(\mathcal{H}(J''))$ such that $G = H[A_{\Delta(\dot{\mathcal{H}}(J)\otimes\mathcal{M})}]$. Since $R(\mathcal{M}, r_0) < r_0$, the matrix H is analytic and bounded in the disk $D(t_{r_0}, R(\mathcal{M}, r_0)^-)$. Hence, by [2, Proposition 2.3.2], the matrix $\mathcal{U}_{G,t_{r_0}}(x) = HWH(t_{r_0})^{-1}$ is also analytic and bounded in the disk $D(t_{r_0}, R(\mathcal{M}, r_0))$. This ends the proof of the lemma in this case.

Case 2: $h > 0$.

Applying Theorem 1.1 to $\mathcal{H}(J') \otimes \mathcal{M}$, there exists a $\mathcal{H}(J'^{p^h})$-differential module $\mathcal{N}_h$ which is a Frobenius antecedent of order h of $\mathcal{H}(J') \otimes \mathcal{M}$. Moreover, $R(\mathcal{N}_h, \rho) < \pi\rho$ for all $\rho \in J'^{p^h}$. Let ${}^hF$ be an associated matrix of $\mathcal{N}_h$. Then, by case 1, there exists $r_0 \in J'$ such that ${}^hF$ is analytic and bounded in the disk $D(t_{r_0}^{p^h}, R(\mathcal{N}_h, r_0^{p^h}))$. The proof of the lemma in this case can be concluded by iteration of Lemma 2.2. □

Proof of Theorem 2.1. By hypothesis, the generic polygon of convergence of $\mathcal{M}$ has only one slope. This slope is a rational number by [5, Theorem 2]. Thus, we may assume there exist $\alpha \in \mathbf{C}_p$ and $\beta \in \mathbf{Q}$ such that $R(\mathcal{M}, r) = |\alpha| r^\beta$ for all $r \in I$.

Let now r be a real in the interior of I. Then, by Lemma 2.3, there exist two reals $r_1, r_2 \in I$ such that $r_1 < r < r_2$ with $\sup_{n\geq 0} \| \frac{G_n}{n!} \|_{r_1} R(\mathcal{M}, r_1)^n < \infty$ and $\sup_{n\geq 0} \| \frac{G_n}{n!} \|_{r_2} R(\mathcal{M}, r_2)^n < \infty$, which are equivalent to $\sup_{n\geq 0} \| \frac{G_n}{n!} \alpha^n x^{n\beta} \|_{r_1} < \infty$ and $\sup_{n\geq 0} \| \frac{G_n}{n!} \alpha^n x^{n\beta} \|_{r_2} < \infty$. Since all the matrices $\alpha^n x^{n\beta} G_n$ have all their entries in $\mathcal{H}[r_1, r_2]$, and for any element $f \in \mathcal{H}([r_1, r_2])$, we have $|f|_r \leq \max(|f|_{r_1}, |f|_{r_2})$, then for any integer $n \geq 0$, we have

$$\begin{aligned}\left\|\frac{G_n}{n!}\right\|_r R(\mathcal{M}, r)^n &\leq \left\|\frac{G_n}{n!}\alpha^n x^{n\beta}\right\|_r \\ &\leq \max\left(\left\|\frac{G_n}{n!}\alpha^n x^{n\beta}\right\|_{r_1}, \left\|\frac{G_n}{n!}\alpha^n x^{n\beta}\right\|_{r_2}\right) \\ &\leq \max\left(\sup_{n\geq 0}\left\|\frac{G_n}{n!}\alpha^n x^{n\beta}\right\|_{r_1}, \sup_{n\geq 0}\left\|\frac{G_n}{n!}\alpha^n x^{n\beta}\right\|_{r_2}\right).\end{aligned}$$

Hence, for

$$\begin{aligned}&\sup_{n\geq 0}\left\|\frac{G_n}{n!}\right\|_r R(\mathcal{M}, r)^n \\ &\quad\leq \max\left(\sup_{n\geq 0}\left\|\frac{G_n}{n!}\alpha^n x^{n\beta}\right\|_{r_1}, \sup_{n\geq 0}\left\|\frac{G_n}{n!}\alpha^n x^{n\beta}\right\|_{r_2}\right) \\ &\quad< \infty.\end{aligned}$$

□

References

[1] F. Baldassarri and B. Chiarellotto, On Christol's theorem. A generalization to systems of PDE's with logarithmic singularities depending upon parameters, in *p-adic methods in number theory and algebraic geometry*, Contemp. Math. **133** (1992), 1–24.

[2] G. Christol, *Modules différentiels et équations différentielles p-adiques*, Queen's Papers in Pure and Applied Mathematics, 66, Queen's Univ., Kingston, ON, 1983.

[3] G. Christol and B. Dwork, Modules différentiels sur des couronnes, Ann. Inst. Fourier (Grenoble) **44** (1994), no. 3, 663–701.

[4] G. Christol and Z. Mebkhout, Sur le théorème de l'indice des équations différentielles p-adiques. III, Ann. of Math. (2) **151** (2000), no. 2, 385–457.

[5] E. Pons, Polygone de convergence d'un module différentiel p-adique, C. R. Acad. Sci. Paris Sér. I Math. **327** (1998), no. 1, 77–80.

[6] P. T. Young, Radii of convergence and index for p-adic differential operators, Trans. Amer. Math. Soc. **333** (1992), no. 2, 769–785.