Branching rules of Dolbeault cohomology groups over indefinite Grassmannian manifolds

By Hideko SEKIGUCHI

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1, Komaba, Meguro-ku, Tokyo 153-8914, Japan

(Communicated by Masaki KASHIWARA, M.J.A., Feb. 14, 2011)

Abstract: We consider a family of singular unitary representations which are realized in Dolbeault cohomology groups over indefinite Grassmannian manifolds, and find a closed formula of irreducible decompositions with respect to reductive symmetric pairs (A_{2n-1}, D_n). The resulting branching rule is a multiplicity-free sum of infinite dimensional, irreducible representations.

Key words: Branching rule; symmetric pair; Penrose transform; singular unitary representation.

1. Introduction. One of basic problems in representation theory is to understand how a given representation is built up from irreducible representations. Typical cases are the decomposition of induced representations (e.g. *Plancherel formulas* for homogeneous spaces) and the decomposition of the restriction (*branching rules, fusion rules, etc.*).

For finite dimensional representations of compact Lie groups, there exist well-known combinatorial algorithms for branching rules such as Littlewood–Richardson's rules and their variants. How about infinite dimensional representations? As of now, there has been no known algorithm for branching rules of general infinite dimensional representations when restricted to noncompact subgroups. To be worse, they may involve infinite multiplicities when the irreducible decomposition contains continuous spectrum even for the restriction with respect to symmetric pairs (see [13,15]).

The concept of "admissible restrictions" of unitary representations of reductive Lie groups singles out an especially nice setting of branching problems, and has opened a promising and concrete analysis after Kobayashi's pioneering and fundamental work [6,7,9,10]. See [2,3,13,15,19,21,22] for some of recent works by Duflo, Gross, Kobayashi, H.-Y. Loke, Ørsted, Speh, Vargas, and Wallach among others in this framework.

In this paper, we highlight a specific branching problem where irreducible unitary representations are realized as Dolbeault cohomology groups on non-compact indefinite Kähler manifolds $Gr_k^+(\mathbf{C}^{n,n})$. Our representations, to be denoted by π_k, are non-tempered irreducible unitary representations of $G = U(n,n)$ for $k < n$. The representation π_k in the case $(n,k) = (2,1)$ appeared in the twister theory by R. Penrose (see [27]). We shall see that the restriction of π_k with respect to the symmetric pair $(G, G') := (U(n,n), SO^*(2n))$ lies in the framework of admissible restrictions.

In the group language, our object may be written symbolically as

$$L \nearrow G \searrow G',$$

where $L \nearrow G$ stands for cohomologically induced representations with respect to a symmetric pair

$$(G, L) := (U(n,n), U(k) \times U(n-k, n))$$

and $G \searrow G'$ stands for the restriction of irreducible unitary representations of G with respect to another symmetric pair

$$(G, G') = (U(n,n), SO^*(2n)).$$

We expand these Dolbeault cohomology groups as a direct sum of the Dolbeault cohomology groups over its complex submanifold defined by $G' = SO^*(2n)$ by using 'normal derivatives' for the cohomology groups. Our technique may be regarded as a cohomological version of the classic technique used for holomorphic sections in Jakobsen–

2000 Mathematics Subject Classification. Primary 22E46; Secondary 05E15, 20G05.

doi: 10.3792/pjaa.87.31

Vergne [5]. The resulting formula of the restriction (see Theorem 2 below) is a generalization of a special case of the Hua–Kostant–Schmid–Kobayashi formula [16,23] for the restriction of holomorphic discrete series representations with respect to semisimple symmetric pairs.

2. Main theorem. We denote by $\mathbf{C}^{n,n}$ the $2n$-dimensional complex vector space $\mathbf{C}^{2n}$ endowed with the indefinite quadratic form of signature (n, n):

$$Q(z) := z_1\overline{z_1} + \cdots + z_n\overline{z_n} - z_{n+1}\overline{z_{n+1}} - \cdots - z_{2n}\overline{z_{2n}}.$$

We say a k-plane α in $\mathbf{C}^{2n}$ is *positive* if the restriction $Q|_\alpha$ is positive definite. The positivity condition forces k to be $k \le n$. Then, the set $Gr_k^+(\mathbf{C}^{n,n})$ of positive k-planes in $\mathbf{C}^{n,n}$ forms a non-empty open subset of the Grassmannian manifold $Gr_k(\mathbf{C}^{2n})$, and consequently it carries a complex structure.

The indefinite unitary group $G = U(n, n)$ is defined to be the subgroup of $GL(2n, \mathbf{C})$ which preserves $Q(z)$. The group G acts biholomorphically and transitively on $Gr_k^+(\mathbf{C}^{n,n})$. We then have a diffeomorphism $Gr_k^+(\mathbf{C}^{n,n}) \simeq G/L(k)$, where

$$L(k) := U(k) \times U(n-k, n)$$

is the isotropy subgroup at the origin $o := \mathbf{C}e_1 + \cdots + \mathbf{C}e_k$. Then the pair $(G, L(k))$ forms a reductive symmetric pair for every $k \le n$. We note that

$$K := L(n) \simeq U(n) \times U(n)$$

is a maximal compact subgroup of G and G/K is realized as a bounded symmetric domain in $M(n, \mathbf{C}) \simeq \mathbf{C}^{n^2}$. For $k < n$, $L(k)$ is non-compact, and the homogeneous space $G/L(k)$ is a $\frac{1}{2}$-Kähler symmetric space in the terminology of M. Berger's classification [1].

For $m \in \mathbf{Z}$, we define a one-dimensional representation by

$$\nu_m : L(k) \to \mathbf{C}^\times, \quad (a, d) \mapsto (\det a)^m.$$

Then, associated to the principal $L(k)$-bundle $G \to Gr_k^+(\mathbf{C}^{n,n})$, we get a G-equivariant holomorphic line bundle

$$\mathcal{L}_m := G \times_{L(k)} (\nu_m, \mathbf{C}) \tag{1}$$

over $G/L(k) \simeq Gr_k^+(\mathbf{C}^{n,n})$. We note that the canonical line bundle over $Gr_k^+(\mathbf{C}^{n,n})$ is isomorphic to $\mathcal{L}_{2n}$ in our normalization.

By the closed range theorem of the $\bar\partial$-operator due to W. Schmid and H.-W. Wong [28], the Dolbeault cohomology group $H^j_{\bar\partial}(Gr_k^+(\mathbf{C}^{n,n}), \mathcal{L}_m)$ carries a Fréchet topology for any j and m, on which we can define a continuous representation of G by translations. Throughout the paper we use the lowercase German letter $\mathfrak{g}$ to denote the complexification of the Lie algebra of a Lie group G. Then the complex Lie algebra $\mathfrak{g}$ also acts on the Fréchet space $H^j_{\bar\partial}(Gr_k^+(\mathbf{C}^{n,n}), \mathcal{L}_m)$.

Let $m = n$. The underlying $(\mathfrak{g}, K)$-module of $H^j_{\bar\partial}(Gr_k^+(\mathbf{C}^{n,n}), \mathcal{L}_n)$ is denoted by $V(j, k, n)$. We have:

Lemma 1.
(1) $V(j, k, n) = 0$ *if* $j \ne k(n-k)$.
(2) $V(j, k, n)$ *is unitarizable if* $j = k(n-k)$.
(3) $V(j, k, n)$ *is a non-zero, irreducible,* $(\mathfrak{g}, K)$-*module if* $j = k(n-k)$.
(4) $V(j, k, n)$ *is* $U(n)$-*admissible where* $U(n)$ *is embedded in* K *as* $g \mapsto (g, g^{-1})$.

Outline of Proof. We note that $V(j, k, n)$ is represented as Zuckerman's derived functor module whose parameter wanders outside the 'good range' but still lies in the 'weakly fair range' in the sense of Vogan [26]. Hence the statements (1) and (2) follow from the general theory of Zuckerman's derived functor modules [25]. The statement (3) is a little more subtle. For this, we use the Beilinson–Bernstein theory for the irreducibility, and the K-type formula for the non-vanishing (see [8,11,24,26]). Finally, we see that $V(j, k, n)$ is $U(n)$-admissible by applying the criterion [7, Theorem 3.2] (or alternatively [12, Theorem 7.4]) for the admissibility of restrictions. □

Next, we set $J_n = \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix}$, and define an involutive automorphism τ of G by

$$\tau(g) := J_n {}^t g^{-1} J_n.$$

Let G' be the identity component of the fixed point group

$$G^\tau := \{g \in G : \tau(g) = g\}.$$

Then G' is isomorphic to $SO^*(2n)$, and $(G, G') = (U(n, n), SO^*(2n))$ forms a reductive symmetric pair.

We write Y for the G'-orbit through the origin o in $Gr_k^+(\mathbf{C}^{n,n})$. It is a closed complex submanifold of $Gr_k^+(\mathbf{C}^{n,n})$, on which G' acts biholomorphically. As a homogeneous space, Y is written as $Y \simeq G'/L'$, where

$$L' := U(k) \times SO^*(2n-2k).$$

For $b = (b_1, \cdots, b_k) \in \mathbf{Z}^k$ such that $b_1 \geq \cdots \geq b_k$, we write $(\pi_b^{U(k)}, V_b)$ for the irreducible finite dimensional representation of $U(k)$ having the highest weight b, and extend it to L' by letting $SO^*(2n-2k)$ act trivially. The resulting representation of L' is denoted by the same letter V_b. Associated to the principal L'-bundle $L' \to G' \to Y$, we define a G'-equivariant holomorphic vector bundle $\mathcal{V}_b$ over $Y \simeq G'/L'$ by

$$\mathcal{V}_b := G' \times_{L'} V_b.$$

Let $K' := K \cap G' \simeq U(n)$, which is a maximal compact subgroup of G'. We note that $K' \simeq U(n)$ is embedded in $K \simeq U(n) \times U(n)$ as $g \mapsto (g, g^{-1})$.

It follows from Lemma 1 (4) that the space of K'-finite vectors in $H_{\bar\partial}^{k(n-k)}(Gr_k^+(\mathbf{C}^{n,n}), \mathcal{L}_n)$ coincides with that of K-finite vectors owing to [10, Proposition 1.6]. This is a dense subspace in the Fréchet space $H_{\bar\partial}^{k(n-k)}(Gr_k^+(\mathbf{C}^{n,n}), \mathcal{L}_n)$, which is stable by the action of the Lie algebra $\mathfrak{g} \simeq \mathfrak{gl}(2n, \mathbf{C})$.

Here is our main theorem:

Theorem 2. *For any $0 \leq k \leq n$, we have an algebraic direct sum decomposition*:

$$H_{\bar\partial}^{k(n-k)}(Gr_k^+(\mathbf{C}^{n,n}), \mathcal{L}_n)_K \simeq \bigoplus_{\substack{a_1 \geq \cdots \geq a_k \geq 0 \\ a_1, \cdots, a_k \in \mathbf{N}}} H_{\bar\partial}^{k(n-k)}(Y, \mathcal{V}_{(2a_1+n, \cdots, 2a_k+n)})_{K'}.$$

The left-hand side is an irreducible $\mathfrak{gl}(2n, \mathbf{C})$-module, whereas the summands in the right-hand side are $\mathfrak{so}(2n, \mathbf{C})$-modules. The resulting branching rule is multiplicity-free. Taking its Hilbert completion, we also obtain the branching rule of the irreducible unitary representation of $G = U(n,n)$ when restricted to the subgroup $G' = SO^*(2n)$.

Detailed proof of Theorem 2 will appear elsewhere.

3. Concluding remarks.

3.1. Size of irreducible summands. For an irreducible representation π of a real reductive group G, we denote by $\mathcal{V}_{\mathfrak{g}}(\pi)$ the associated variety of its underlying $(\mathfrak{g}, K)$-module. The dimension of $\mathcal{V}_{\mathfrak{g}}(\pi)$ equals the Gelfand–Kirillov dimension of π, and $\mathcal{V}_{\mathfrak{g}}(\pi)$ may be regarded as a coarse measure of the 'size' of π.

Suppose G' is a reductive subgroup of G, and we take a maximal compact subgroup K of G such that $K' := K \cap G'$ is a maximal compact subgroup of G'. Then, by the general theory due to Kobayashi [10], the associated varieties $\mathcal{V}_{\mathfrak{g}'}(\tau)$ are all the same for irreducible $(\mathfrak{g}', K')$-modules τ such that $\mathrm{Hom}_{\mathfrak{g}', K'}(\tau, \pi|_{(\mathfrak{g}', K')}) \neq 0$, and furthermore, they have the following lower bound:

$$\mathrm{pr}_{\mathfrak{g} \to \mathfrak{g}'}(\mathcal{V}_{\mathfrak{g}}(\pi)) \subset \mathcal{V}_{\mathfrak{g}'}(\tau), \tag{2}$$

where $\mathrm{pr}_{\mathfrak{g} \to \mathfrak{g}'} : \mathfrak{g}^* \to (\mathfrak{g}')^*$ is the restriction map. It follows from the explicit formula in Theorem 2 that the equality holds in (2) in our setting.

3.2. Finite dimensional analogues. Finite dimensional analogues of our branching rules have been found quite recently by combinatorial methods by S. Okada [20]. In other words, Theorem 2 may be thought of as their infinite dimensional generalization.

In order to compare them with Theorem 2, we consider the following replacement:

$$Gr_k^+(\mathbf{C}^{n,n}) \Longrightarrow Gr_k(\mathbf{C}^{2n})$$
$$U(n,n) \Longrightarrow U(2n)$$
$$SO^*(2n) \Longrightarrow SO(2n).$$

Correspondingly, we replace the infinite dimensional representations of $U(n,n)$ on $H_{\bar\partial}^{k(n-k)}(Gr_k^+(\mathbf{C}^{n,n}), \mathcal{L}_n)$ by the finite dimensional representations of $U(2n)$ on $H^0(Gr_k(\mathbf{C}^{2n}), \mathcal{L}_{-m})$ which are irreducible by the classical Borel–Weil theorem. Then, instead of the setting of Theorem 2 (i.e. the restriction $U(n,n) \downarrow SO^*(2n)$), we consider the restriction $U(2n) \downarrow SO(2n)$. Since $SO(2n)$ acts on $Gr_k(\mathbf{C}^{2n})$ in a strongly visible fashion [17], we see that the spaces $H^0(Gr_k(\mathbf{C}^{2n}), \mathcal{L}_{-m})$ decompose as a multiplicity-free sum of irreducible finite dimensional representations of $SO(2n)$ for any $0 < k < 2n$ and m owing to Kobayashi's multiplicity-free theorem [16, Theorem E] (see also [14]). Explicit branching rules in the special case $k = n$ were previously obtained by Okada [20, Theorem 2.6] by using minor summation formulas of Ishikawa and Wakayama [4]. (However, Okada's formula loc. cit. involves some minor misprints.) Alternatively, the same branching rules could be computed by using Richardson–Littlewood, or Koike–Terada's algorithm [18].

3.3. Case of discrete series representations. If π is a discrete series representation of G, then any irreducible, discrete summand τ of the restriction $\pi|_{G'}$ becomes a discrete series representation of G' [12]. In this case, Duflo and Vargas [2] announced combinatorial formulas for its irreducible decomposition in the framework of admissible restrictions

[7,9,10] by using the Heckman–Duflo–Vergne formula for the quantization of coadjoint orbits. Their method gives an alternative approach to the case $k = n$, but not to the case $k < n$ where the representations π_k are not tempered.

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