Determinant formulas for zeta functions for real abelian function fields

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Abstract: In this paper, we will give determinant formulas of zeta functions for real abelian extensions over a rational functions field with one variable. By a class number formula, our formula can be regard as a generalization of determinant formulas of class numbers.

Key words: Zeta functions; cyclotomic function fields.

1. Introduction. Let k be a field of rational functions over a finite field $\mathbf{F}_q$ with q elements. Fix a generator T of k, and let $A = \mathbf{F}_q[T]$ be the polynomial subring of k. For a monic polynomial $m \in A$, let Λ_m be the set of m-torsion points of the Carlitz module (see section 2). Let $K_m = k(\Lambda_m)$. The function field K_m is called the m-th cyclotomic function field, which is an analogue of cyclotomic field over $\mathbf{Q}$.

Let $P \in A$ be a monic irreducible polynomial. In the late 1990s, Rosen gave a determinant formula for the relative class number of K_P (cf. [Ro1]), which is regarded as an analogue of the classical Maillet determinant. Recently, Ahn, Bae, Choi, and Jung generalized the Rosen's formula to any subfield of cyclotomic function fields with arbitrary conductor (cf. [A-B-J, A-C-J]).

In this paper, we will extend these formulas of class numbers to those of zeta functions. For a global function field M over $\mathbf{F}_q$, define the zeta function by

$$\zeta(s, M) = \prod_{\mathcal{P}} \left(1 - \frac{1}{\mathcal{N}\mathcal{P}^s}\right)^{-1},$$

where the product runs over all primes of M, and $\mathcal{N}\mathcal{P}$ is the number of elements of the residue class field of $\mathcal{P}$. By the standard facts about the zeta function (cf. [Ro2] chapter 5), there is a polynomial $Z_M(X) \in \mathbf{Z}[X]$ such that

$$\zeta(s, M) = \frac{Z_M(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}. \tag{1}$$

In the previous paper [Sh], the author constructed the determinant formula for $Z_{K_m^+}(X)$, where K_m^+ is the maximal real subfield K_m. Our goal of this paper is to generalize this result to any real subfield of K_m (see Theorem 3.1).

Let h_M be the class number of M, which is the order of the divisor class group of degree 0. Since $Z_M(1) = h_M$, our formula derives a class number formula (see Corollary 3.1).

2. Preparations. In this section, we review definitions and basic properties of cyclotomic function fields, and Dirichlet characters. For more information, see [Ha,Ro2,Wa]. Let us denote by $\bar{k}$ an algebraic closure of k. For $x \in \bar{k}$ and $m \in A$, we define the following action:

$$m * x = m(\varphi + \mu)(x),$$

where φ, μ are $\mathbf{F}_q$-linear map defined by

$$\varphi : \bar{k} \longrightarrow \bar{k} \quad (x \mapsto x^q),$$
$$\mu : \bar{k} \longrightarrow \bar{k} \quad (x \mapsto Tx).$$

By the above actions, $\bar{k}$ becomes an A-module, which is called the Carlitz module. Let Λ_m be the set of all x satisfying $m * x = 0$. Let $K_m = k(\Lambda_m)$. The field K_m is called the m-th cyclotomic function field. It is well-known that K_m/k is a finite Galois extension, and its Galois group $\mathrm{Gal}(K_m/k)$ is isomorphic to G_m, where G_m is the unit group of the quotient ring A/mA. Put

$$\widetilde{K} = \bigcup_{m:\text{monic}} K_m,$$

where m runs through all monic polynomials of A. For a finite extension M over k contained in $\widetilde{K}$, the conductor of M is defined as the monic polynomial m such that K_m is the smallest cyclotomic function field containing M. Let H_M be the subgroup of G_m corresponding to M. We regard $\mathbf{F}_q^\times \subseteq G_m$. We shall call M real if $\mathbf{F}_q^\times \subseteq H_M$. Otherwise, we shall call M imaginary.

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Let P_∞ be the unique prime of k which corresponds to the valuation ord_∞ with $\mathrm{ord}_\infty(T) < 0$. We denote by k_∞ the completion of k by ord_∞. Then we see that $M \subseteq k_\infty$ if and only if M is real.

Next, we will give basic facts about Dirichlet characters. For a monic polynomial $m \in A$, let X_m be the group of all primitive Dirichlet characters modulo m. Denote by $\mathbf{D}$ the group of all primitive Dirichlet characters (i.e. $\mathbf{D} = \bigcup_{m:\text{monic}} X_m$). Then, by the same argument as in the case of number field, we have a one-to-one correspondence between finite subgroups of $\mathbf{D}$ and finite subextension of $\widetilde{K}/k$ (cf. [Wa] chapter 3). In particular, X_m corresponds to K_m.

Let M be a real abelian extension over k with conductor m. Let X_M be the subgroup of $\mathbf{D}$ corresponding to M. For $\chi \in X_M$, define an L-function by

$$L(s,\chi) = \prod_P \left(1 - \frac{\chi(P)}{\mathcal{N}P^s}\right)^{-1},$$

where P runs through all monic irreducible polynomials of A. By the same argument as in the case of number fields, we have the following decomposition by L-functions:

$$\prod_{\mathcal{P}:\text{finite}} \left(1 - \frac{1}{\mathcal{N}\mathcal{P}^s}\right)^{-1} = \prod_{\chi \in X_M} L(s,\chi),$$

where the product of the left hand runs through primes of M not dividing P_∞. Since M is real, the prime P_∞ totally splits in M/k. Hence

$$\zeta(s,M) = \left\{\prod_{\chi\in X_M} L(s,\chi)\right\}(1-q^{-s})^{-[M:k]}.$$

Let χ_0 be the trivial character. Then we see that $L(s,\chi_0) = 1/(1-q^{1-s})$. Hence, by equation (1), we have

$$Z_M(q^{-s}) = \left\{\prod_{\substack{\chi\in X_M\\ \chi\neq\chi_0}} L(s,\chi)\right\}(1-q^{-s})^{1-[M:k]}. \tag{2}$$

We will use the above equation (2) to prove our determinant formula.

3. Determinant formulas. Let M be a real abelian extension over k with conductor m. Our goal in this section is to construct a determinant formula for $Z_M(X)$. To do this, we first give some notations.

Let H_M be the subgroup of G_m corresponding to M. Let X_M be the subgroup of $\mathbf{D}$ corresponding to M. For $\alpha \in G_m$, there is the unique element $r_\alpha \in A$ such that

$$\begin{aligned} r_\alpha &= a_n T^n + a_{n-1}T^{n-1} + \cdots + a_0 \\ &\qquad (n < \deg m,\ a_n \neq 0), \\ r_\alpha &\equiv \alpha \bmod m. \end{aligned}$$

Then we define functions Deg and L over G_m as

$$\mathrm{Deg}(\alpha) = n, \quad \mathrm{L}(\alpha) = a_n \in \mathbf{F}_q^\times.$$

We notice that Deg is a function over $G_m/\mathbf{F}_q^\times$. Put

$$\Delta_M = \{\alpha \in H_M \mid \mathrm{L}(\alpha) = 1\}.$$

Then we see that $H_M = \mathbf{F}_q^\times \times \Delta_M$. Next, we define

$$F_\alpha(X) = \sum_{\beta\in\alpha\Delta_M} X^{\mathrm{Deg}(\beta)}$$

for $\alpha \in G_m$. Then we can easily check that $F_{\alpha_1}(X) = F_{\alpha_2}(X)$ if $\alpha_1 H_M = \alpha_2 H_M$. Let $N_M = [M:k]-1$, and let $\alpha_0 = 1, \alpha_1, \ldots, \alpha_{N_M}$ be a complete system of representatives for G_m/H_M with $\mathrm{L}(\alpha) = 1$. For $i,j = 1,2,\ldots,N_M$, put

$$F_{ij}(X) = (F_{\alpha_i\alpha_j^{-1}}(X) - F_{\alpha_i}(X))/(1-X).$$

Define the matrix $E_M(X)$ by

$$E_M(X) = (F_{ij}(X))_{i,j=1,2,\ldots,N_M}.$$

Then we have the following determinant formula for zeta functions.

Theorem 3.1. *In the above notations, we have*

$$\det E_M(X) = J_M(X)Z_M(X).$$

Here $J_M(X)$ is a polynomial defined by

$$J_M(X) = \prod_{\substack{\chi\in X_M\\ \chi\neq\chi_0}} \prod_{Q\mid m} (1-\chi(Q)X^{\deg Q}),$$

where the second product runs through all irreducible monic polynomials dividing m.

Remark 3.1. By the same argument in Proposition 3.1 in [Sh], we have

$$J_M(X) = \prod_{Q\mid m} \frac{(1-X^{f_Q \deg Q})^{g_Q}}{(1-X^{\deg Q})}$$

where f_Q are the residue class degrees of Q in M/k, and g_Q are the numbers of primes in M over Q. Hence we see that $J_M(X)$ is a polynomial of integral coefficients.

Now we give the proof of Theorem 3.1.

Proof. For $\chi \in X_M$, denote by f_χ the conductor of χ. We put $\tilde{\chi} = \chi \circ \pi_\chi$, where $\pi_\chi : G_m \to G_{f_\chi}$ is a natural homomorphism. Then $\{\tilde{\chi} : \chi \in X_M\}$ is the character group of G_m/H_M. For $\chi \in X_M$, we see that

$$L(s, \tilde{\chi}) = L(s, \chi) \cdot \prod_{Q|m} (1 - \chi(Q) q^{-s \deg Q}).$$

Hence we use equation (2) to obtain

$$\prod_{\substack{\chi \in X_M \\ \chi \neq \chi_0}} L(s, \tilde{\chi}) = Z_M(q^{-s}) J_M(q^{-s}) (1 - q^{-s})^{N_M}. \tag{3}$$

Let $\chi \in X_M$ be a non-trivial character. Then we see that

$$L(s, \tilde{\chi}) = \sum_{\substack{\alpha \in G_m \\ \mathrm{L}(\alpha) = 1}} \tilde{\chi}(\alpha) q^{-\mathrm{Deg}(\alpha) s}.$$

(cf. [Ro2] chapter 4). Noting that $\tilde{\chi}$ is a character of G_m/H_M, we have

$$\begin{aligned} L(s, \tilde{\chi}) &= \sum_{i=0}^{N_M} \sum_{\beta \in \alpha_i \Delta_M} \tilde{\chi}(\beta) q^{-\mathrm{Deg}(\beta) s} \\ &= \sum_{i=0}^{N_M} \tilde{\chi}(\alpha_i) F_{\alpha_i}(q^{-s}). \end{aligned}$$

Fix $s \in \mathbf{C}$. We notice that $F_\alpha(q^{-s})$ is a function over G_m/H_M. By applying the Frobenius determinant formula for the group G_m/H_M and the function $F_\alpha(q^{-s})$, we obtain

$$\begin{aligned} &\prod_{\substack{\chi \in X_M \\ \chi \neq \chi_0}} L(s, \tilde{\chi}) \\ &= \prod_{\substack{\chi \in X_M \\ \chi \neq \chi_0}} \sum_{i=0}^{N_M} \tilde{\chi}(\alpha_i) F_{\alpha_i}(q^{-s}) \\ &= \det(F_{\alpha_i \alpha_j^{-1}}(q^{-s}) - F_{\alpha_i}(q^{-s}))_{i,j=1,2,\ldots,N_M} \end{aligned}$$

(cf. [Wa] Lemma 5.26). By equation (3), we have

$$\det E_M(q^{-s}) = Z_M(q^{-s}) J_M(q^{-s}).$$

This completes the proof of Theorem 3.1. □

By an analytic class number formula, we have $Z_M(1) = h_M$. Hence our formula leads the following class number formula.

Corollary 3.1. *In the above notations, we have*

$$\det\left(\sum_{\beta \in \alpha_i \Delta_M} \mathrm{Deg}(\beta) - \sum_{\beta \in \alpha_i \alpha_j^{-1} \Delta_M} \mathrm{Deg}(\beta) \right)_{i,j=1,2,\ldots,N_M} = h_M R_M,$$

where R_M is the integer defined by

$$R_M = \begin{cases} \prod_{Q|m} f_Q & \text{if } g_Q = 1 \text{ for every prime } Q \\ & \text{dividing } m, \\ 0 & \text{otherwise,} \end{cases}$$

where f_Q is the residue class degree of Q in M/k, and g_Q is the number of primes in M over Q.

Remark 3.2. The above class number formula was first given by Ahn, Bae, and Jung (cf. [A-B-J] Proposition 3.3).

Example 3.1. Let $q = 3$, $m = T^3$. Put

$$H = \{1, T+1, T^2 + 2T + 1\} \times \mathbf{F}_3^\times \subset G_m.$$

Let M be the intermediate field of K_m/k corresponding to H. Then we see that $\Delta_M = \{1, T+1, T^2 + 2T + 1\}$. Put

$$\alpha_0 = 1,\ \alpha_1 = T + 2,\ \alpha_2 = T^2 + T + 1.$$

Then we have

$$E_M(X) = \begin{pmatrix} 1 + X & -X \\ X & 1 + 2X \end{pmatrix}, \quad J_M(X) = 1.$$

By applying Theorem 3.1, we have $Z_M(X) = \det E_M(X) = 1 + 3X + 3X^2$. The class number h_M is $Z_M(1) = 7$.

References

[A-B-J] H. Jung, S. Bae and J. Ahn, Determinant formulas for class numbers in function fields, Math. Comp. **74** (2005), no. 250, 953–965.

[A-C-J] J. Ahn, S. Choi and H. Jung, Class number formulae in the form of a product of determinants in function fields, J. Aust. Math. Soc. **78** (2005), no. 2, 227–238.

[Ha] D. R. Hayes, Explicit class field theory for rational function fields, Trans. Amer. Math. Soc. **189** (1974), 77–91.

[Ro1] M. Rosen, A note on the relative class number in function fields, Proc. Amer. Math. Soc. **125** (1997), no. 5, 1299–1303.

[Ro2] M. Rosen, *Number theory in function fields*, Graduate Texts in Mathematics, 210, Springer, New York, 2002.

[Sh] D. Shiomi, A determinant formula for congruence zeta functions of maximal real cyclotomic function fields, Acta Arith. **138** (2009), no. 3, 259–268.

[Wa] L. C. Washington, *Introduction to cyclotomic fields*, Graduate Texts in Mathematics, 83, Springer, New York, 1982.