Remark on dynamical Morse inequality

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Abstract: We solve a transversality problem relating to Bertelson-Gromov's "dynamical Morse inequality".

Key words: Critical point; Morse function; Betti-number entropy.

1. Introduction. Bertelson-Gromov proposed a study of "dynamical Morse inequality" in [2]. It is a new kind of Morse theory in (asymptotically) infinite dimensional situations. The authors think that the paper [2] opened a way to a fruitful new research area. The purpose of this paper is to solve a transversality problem relating to [2].

Let X be a compact connected smooth manifold of dimension ≥ 1, and let $f : X \times X \to \mathbf{R}$ be a smooth function. For $n = 1, 2, 3, \dots,$ we define $S_n(f) : X^{n+1} \to \mathbf{R}$ by

$$S_n(f)(x_0, x_1, \dots, x_n) := \sum_{i=0}^{n-1} f(x_i, x_{i+1}). \tag{1}$$

The study of this kind of functions was proposed by Bertelson-Gromov [2]. (See also Bertelson [1].) The "physical" meaning of $S_n(f)$ is as follows. We consider a crystal which consists of n particles in a line. Suppose that the state of each particle is described by the manifold X and that each particle interacts with the next one by the potential function $f(x, y)$. Then the critical points of $S_n(f)$ correspond to the stationary states of the crystal.

Let c be a real number, and let δ be a positive real number. We define $N_n(c, \delta)$ as the number of critical points p of $S_n(f)$ with $c - \delta < S_n(f)(p)/n < c + \delta$:

$$N_n(c, \delta) := \sharp\left\{ p \in X^{n+1} \mid (dS_n(f))_p = 0, \; c - \delta < \frac{S_n(f)(p)}{n} < c + \delta \right\}.$$

We set

$$N(c) := \lim_{\delta \to 0}\left(\liminf_{n \to \infty} \frac{\log N_n(c, \delta)}{n} \right).$$

Recall that a smooth function on a manifold is called a Morse function if all its critical points are non-degenerate. Bertelson-Gromov [2] proved the following "dynamical Morse inequality". (See [2, Remark 8.2] and [1, p. 156, Remark].)

Theorem 1.1 (Bertelson-Gromov)**.** *Suppose the following*

$$\textit{All } S_n(f) : X^{n+1} \to \mathbf{R} \ (n \geq 1) \textit{ are Morse functions.} \tag{2}$$

Then for any $c \in \mathbf{R}$

$$N(c) \geq b(c). \tag{3}$$

Here $b(c)$ *is the "Betti-number entropy" introduced in* [2]. ($b(c)$ *depends on* f.) *We review its definition in Appendix A.*

The function $b(c)$ is concave, and there exists $c \in \mathbf{R}$ such that $b(c) > 0$ [2, Proposition 9.2, Proposition 10.1].

Theorem 1.1 rises the following natural question: How common is the condition (2) for smooth functions? The main issue of this note is to give an affirmative answer to this question. Notice that the answer is not apparent because of the symmetry of the function $S_n(f)$. For example, the value

$$\begin{aligned} &S_n(f)(x, \dots, x, y_1, \dots, y_m, x, \dots, x) \\ &\quad = f(x, y_1) + f(y_m, x) + (n - m - 1) f(x, x) \\ &\qquad + \sum_{i=1}^{m-1} f(y_i, y_{i+1}) \end{aligned}$$

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does not depend on the number of x's before the sequence of $y_1, \ldots, y_m$ appears. So the standard arguments to show the prevalence of Morse functions (e.g. Guillemin-Pollack [3, Chapter 1, Section 7], Hirsch [4, Chapter 6, Section 1]) do not work.

Let $\mathcal{C}^\infty(X \times X)$ be the space of all (real valued) $\mathcal{C}^\infty$-functions in $X \times X$. $\mathcal{C}^\infty(X \times X)$ is endowed with the topology of $\mathcal{C}^\infty$-convergence. A subset $U \subset \mathcal{C}^\infty(X \times X)$ is said to be residual if it contains a countable intersection of open dense subsets of $\mathcal{C}^\infty(X \times X)$. The main result of this paper is the following

Theorem 1.2. *The set of all functions $f \in \mathcal{C}^\infty(X \times X)$ satisfying the condition* (2) *is a residual subset of $\mathcal{C}^\infty(X \times X)$.*

Let $\mathcal{C}_s^\infty(X \times X)$ be the space of all $f \in \mathcal{C}^\infty(X \times X)$ satisfying the symmetric relation $f(x, y) = f(y, x)$ for all $x, y \in X$. This is a closed subspace of $\mathcal{C}^\infty(X \times X)$. If we consider X^{n+1} as the "configuration space of a crystal" as we explained before, then it is natural to suppose that the "potential function" f is symmetric. So we think that the following result is also interesting.

Theorem 1.3. *The set of all functions $f \in \mathcal{C}_s^\infty(X \times X)$ satisfying* (2) *is a residual subset of $\mathcal{C}_s^\infty(X \times X)$.*

2. Proof of Theorems 1.2 and 1.3. In this section we assume that the closed manifold X is smoothly embedded into the Euclidean space $\mathbf{R}^N$. For $n \geq 1$, let $\mathbf{P}_n$ be the set of all partitions of $\{0, 1, 2, \ldots, n\}$. For $\sigma = \{P_1, P_2, \ldots, P_l\} \in \mathbf{P}_n$, we set $|\sigma| = l$ and $\sigma(i) = P_j$ for $i \in P_j$, $(i = 0, 1, 2, \ldots, n)$. For example, if $\sigma = \{\{0\}, \{1, 3\}, \{2\}\} \in \mathbf{P}_3$, then $|\sigma| = 3$ and $\sigma(0) = \{0\}$, $\sigma(1) = \sigma(3) = \{1, 3\}$, $\sigma(2) = \{2\}$. We define an order on $\mathbf{P}_n$ as follows: For $\sigma, \tau \in \mathbf{P}_n$, $\tau \geq \sigma$ if we have $\tau(i) \supset \sigma(i)$ for all $i = 0, 1, \ldots, n$. (This means that σ is a subdivision of τ.) The maximum partition with respect to this ordering is $\{\{0, 1, 2, \ldots, n\}\}$, and the minimum partition is $\{\{0\}, \{1\}, \ldots, \{n\}\}$.

For $\sigma \in \mathbf{P}_n$, we set

$$\begin{aligned} X_\sigma &:= \{(x_0, x_1, \ldots, x_n) \in X^{n+1} \mid x_i = x_j \\ &\qquad \text{if } \sigma(i) = \sigma(j)\}, \\ \mathbf{R}_\sigma^N &:= \{(v_0, v_1, \ldots, v_n) \in (\mathbf{R}^N)^{n+1} \mid v_i = v_j \\ &\qquad \text{if } \sigma(i) = \sigma(j)\}. \end{aligned}$$

We have $X_\sigma \subset \mathbf{R}_\sigma^N$. If $\tau \geq \sigma$, then $X_\tau \subset X_\sigma$ and $\mathbf{R}_\tau^N \subset \mathbf{R}_\sigma^N$. We set

$$\Sigma_\sigma := \bigcup_{\tau \gneqq \sigma} X_\tau.$$

Here τ runs over all partitions in $\mathbf{P}_n$ strictly greater than σ. We have $\Sigma_\sigma \subset X_\sigma$.

Remark 2.1. (i) For $\boldsymbol{x} = (x_0, x_1, \ldots, x_n) \in X^{n+1}$, we have $\boldsymbol{x} \in X_\sigma \setminus \Sigma_\sigma$ if and only if the following condition is satisfied: "$x_i = x_j \Leftrightarrow \sigma(i) = \sigma(j)$".
(ii) $X^{n+1} = \bigcup_{\sigma \in \mathbf{P}_n} (X_\sigma \setminus \Sigma_\sigma)$.
(iii) The pair $(\mathbf{R}_\sigma^N, X_\sigma)$ is diffeomorphic to the pair $((\mathbf{R}^N)^{|\sigma|}, X^{|\sigma|})$.

For $f \in \mathcal{C}^\infty(X \times X)$, we define $S_n(f) \in \mathcal{C}^\infty(X^{n+1})$ by (1). For each (fixed) $n \geq 1$, the set $\{f \in \mathcal{C}^\infty(X \times X) \mid S_n(f)$ is a Morse function$\}$ is obviously open in $\mathcal{C}^\infty(X \times X)$. (A similar statement for $\mathcal{C}_s^\infty(X \times X)$ is also true.) Hence Theorems 1.2 and 1.3 follow from the following

Theorem 2.2. *Fix $n \geq 1$. Every $f \in \mathcal{C}^\infty(X \times X)$ can be approximated arbitrarily well* (*in the $\mathcal{C}^\infty$-topology*) *by $g \in \mathcal{C}^\infty(X \times X)$ such that $S_n(g)$ is a Morse function. Moreover, if f is symmetric* (*i.e. $f(x, y) = f(y, x)$ for all $x, y \in X$*), *then we can choose a symmetric approximation g.*

For a while we will prepare some preliminary results for proving this theorem. In the rest of this section we fix $n \geq 1$. First recall the following well-known result (see Guillemin-Pollack [3, p. 43]).

Proposition 2.3. *Let M be a closed manifold embedded in $\mathbf{R}^m$, and let $f : M \to \mathbf{R}$ be a smooth function. Fix $x_0 \in \mathbf{R}^m$. Then for almost every $\alpha \in \mathbf{R}^m$, the function*

$$M \ni x \mapsto f(x) + \langle \alpha, x - x_0 \rangle \in \mathbf{R}$$

is a Morse function. Here $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathbf{R}^m$.

We will also need the following (well-known, we believe).

Lemma 2.4. *Let M be a closed manifold embedded in $\mathbf{R}^m$, and let $f : M \to \mathbf{R}$ be a smooth function. Let $p = (p_1, \ldots, p_m) \in M$ be a critical point of f. Let $a_1, \ldots, a_m$ be positive numbers. Then for all but finitely many $c \in \mathbf{R}$, the point p is a non-degenerate critical point of the following function*:

$$g_c : M \ni x \mapsto f(x) + c \sum_{i=1}^{m} a_i |x_i - p_i|^2 \in \mathbf{R}.$$

Proof. First note that the following fact: Let A and B be two matrices of the same degree, and suppose B is regular. Then $A+cB$ is also regular for $c \gg 1$. Hence $\det(A+cB)$ is not identically zero as the polynomial of c. So it has only finitely many zeros. Then $A+cB$ is regular for all but finitely many $c \in \mathbf{R}$.

We can assume $p=0$. Let $\varphi=(\varphi_1,\dots,\varphi_m):\mathbf{R}^k \to M \subset \mathbf{R}^m$ $(\varphi(0)=0)$ be a local coordinate around $0 \in M$. We have $g_c \circ \varphi(y) = f\circ\varphi(y) + c\sum_{i=1}^m a_i(\varphi_i(y))^2$. Then

$$\frac{\partial^2 g_c\circ\varphi}{\partial y_\alpha \partial y_\beta}(0) = \frac{\partial^2 f\circ\varphi}{\partial y_\alpha \partial y_\beta}(0) + 2c\sum_{i=1}^m a_i \frac{\partial\varphi_i}{\partial y_\alpha}(0)\frac{\partial\varphi_i}{\partial y_\beta}(0).$$

It is easy to see that the symmetric matrix $(\sum_i a_i \cdot \partial\varphi_i(0)/\partial y_\alpha \cdot \partial\varphi_i(0)/\partial y_\beta)_{\alpha,\beta}$ is positive definite and hence regular. Hence the desired result follows from the above remark. $\square$

For $\boldsymbol{p}=(p_0,p_1,\dots,p_n)\in X^{n+1}$, we put

$$r(\boldsymbol{p}) := \min\{|p_i-p_j| \mid p_i \neq p_j\},$$
$$U_{\boldsymbol{p}} := \{\boldsymbol{x}\in X^{n+1} \mid |\boldsymbol{x}-\boldsymbol{p}| < r(\boldsymbol{p})/3\}.$$

When $p_0=p_1=\cdots=p_n$, we set $r(\boldsymbol{p})=+\infty$ and $U_{\boldsymbol{p}}=X^{n+1}$. Let χ be a $\mathcal{C}^\infty$-function on $\mathbf{R}$ such that $\chi=1$ on $[0,1/3]$ and $\chi=0$ on $[2/3,+\infty)$. For $\boldsymbol{p}=(p_0,p_1,\dots,p_n)\in X^{n+1}$ and $j=0,1,\dots,n$, we set $\chi_{\boldsymbol{p},j}(x) := \chi(|x-p_j|/r(\boldsymbol{p}))$ for $x\in X$. If $p_0=p_1=\cdots=p_n$, then we set $\chi_{\boldsymbol{p},j}\equiv 1$.

Lemma 2.5. *Let $\sigma\in\mathbf{P}_n$. For $\boldsymbol{p}\in X_\sigma\setminus\Sigma_\sigma$ and $\boldsymbol{x}=(x_0,x_1,\dots,x_n)\in U_{\boldsymbol{p}}$,*

$$\chi_{\boldsymbol{p},j}(x_i) = \begin{cases} 1 & \text{if } \sigma(i)=\sigma(j),\\ 0 & \text{if } \sigma(i)\neq\sigma(j).\end{cases}$$

Proof. If $\sigma(i)=\sigma(j)$, then $p_i=p_j$. So $|x_i-p_j|=|x_i-p_i|\le|\boldsymbol{x}-\boldsymbol{p}|<r(\boldsymbol{p})/3$. If $\sigma(i)\neq\sigma(j)$, then $p_i\neq p_j$. So $|x_i-p_j|\ge|p_i-p_j|-|x_i-p_i|\ge 2r(\boldsymbol{p})/3$. $\square$

For $i=0,1,2,\dots,n$, we set

$$\mu(i) = \begin{cases} 1 & i=0,n,\\ 2 & \text{otherwise.}\end{cases}$$

Lemma 2.6. *Let $\sigma\in\mathbf{P}_n$, $\boldsymbol{p}=(p_0,p_1,\dots,p_n)\in X_\sigma\setminus\Sigma_\sigma$ and $\boldsymbol{\alpha}=(\alpha_0,\alpha_1,\dots,\alpha_n)\in\mathbf{R}^N_\sigma$. Then there is a symmetric $f_{\boldsymbol{p},\boldsymbol{\alpha}}\in\mathcal{C}^\infty(X\times X)$ such that $S_n(f_{\boldsymbol{p},\boldsymbol{\alpha}})(\boldsymbol{x})=\langle\boldsymbol{\alpha},\boldsymbol{x}-\boldsymbol{p}\rangle$ for all $\boldsymbol{x}\in U_{\boldsymbol{p}}\cap X_\sigma$.*

Proof. We define $h\in\mathcal{C}^\infty(X)$ by

$$h(x) := \sum_{j=0}^n\left(\sum_{k\in\sigma(j)}\mu(k)\right)^{-1}\langle\alpha_j,\chi_{\boldsymbol{p},j}(x)(x-p_j)\rangle.$$

Put $f_{\boldsymbol{p},\boldsymbol{\alpha}}(x,y) := h(x)+h(y)$. For $\boldsymbol{x}=(x_0,x_1,\dots,x_n)\in U_{\boldsymbol{p}}\cap X_\sigma$,

$$\begin{aligned} &S_n(f_{\boldsymbol{p},\boldsymbol{\alpha}})(\boldsymbol{x})\\ &= \sum_{i=0}^{n-1}(h(x_i)+h(x_{i+1})) = \sum_{i=0}^n\mu(i)h(x_i)\\ &= \sum_{0\le i,j\le n}\mu(i)\left\{\sum_{k\in\sigma(j)}\mu(k)\right\}^{-1}\langle\alpha_j,\chi_{\boldsymbol{p},j}(x_i)(x_i-p_j)\rangle\\ &= \sum_{j=0}^n\left(\sum_{k\in\sigma(j)}\mu(k)\right)^{-1}\left(\sum_{i\in\sigma(j)}\mu(i)\langle\alpha_j,x_j-p_j\rangle\right)\\ &\quad(\text{by Lemma 2.5 and } x_i=x_j \text{ for } i\in\sigma(j))\\ &= \sum_{j=0}^n\langle\alpha_j,x_j-p_j\rangle = \langle\boldsymbol{\alpha},\boldsymbol{x}-\boldsymbol{p}\rangle. \end{aligned}$$

$\square$

Lemma 2.7. *For $\boldsymbol{p}=(p_0,p_1,\dots,p_n)\in X^{n+1}$, there is a symmetric $g_{\boldsymbol{p}}\in\mathcal{C}^\infty(X\times X)$ such that $S_n(g_{\boldsymbol{p}})(\boldsymbol{x})=\sum_{i=0}^n\mu(i)|x_i-p_i|^2$ for all $\boldsymbol{x}=(x_0,\dots,x_n)\in U_{\boldsymbol{p}}$.*

Proof. Choose $\sigma\in\mathbf{P}_n$ such that $\boldsymbol{p}\in X_\sigma\setminus\Sigma_\sigma$. We define $h\in\mathcal{C}^\infty(X)$ by

$$h(x) := \sum_{j=0}^n\frac{\chi_{\boldsymbol{p},j}(x)}{\sharp\sigma(j)}|x-p_j|^2.$$

Set $g_{\boldsymbol{p}}(x,y) := h(x)+h(y)$. For $\boldsymbol{x}=(x_0,x_1,\dots,x_n)\in U_{\boldsymbol{p}}$,

$$\begin{aligned} &S_n(g_{\boldsymbol{p}})(\boldsymbol{x})\\ &= \sum_{i=0}^{n-1}(h(x_i)+h(x_{i+1})) = \sum_{i=0}^n\mu(i)h(x_i)\\ &= \sum_{0\le i,j\le n}\mu(i)\frac{\chi_{\boldsymbol{p},j}(x_i)}{\sharp\sigma(j)}|x_i-p_j|^2\\ &= \sum_{i=0}^n\mu(i)\sum_{j\in\sigma(i)}\frac{1}{\sharp\sigma(i)}|x_i-p_j|^2 \quad (\text{by Lemma 2.5})\\ &= \sum_{i=0}^n\mu(i)|x_i-p_i|^2 \quad (p_j=p_i \text{ for } j\in\sigma(i)). \end{aligned}$$

$\square$

Let M be a manifold, and let $f:M\to\mathbf{R}$ be a smooth function. We define $C(f)$ as the set of all critical points of f, and $C_*(f)$ as the set of all degenerate critical points of f.

Lemma 2.8. *Let $\sigma\in\mathbf{P}_n$, and let $K\subset X_\sigma$ be a compact set. Let $f\in\mathcal{C}^\infty(X\times X)$. Suppose $C_*(S_n(f)|_{X_\sigma})\cap K=\emptyset$. Then f can be approximated arbitrarily well by $g\in\mathcal{C}^\infty(X\times X)$ such that $C_*(S_n(g))\cap K=\emptyset$. If f is symmetric, then we can choose a symmetric approximation g.*

Proof. All critical points of $S_n(f)|_{X_\sigma}$ in K are isolated in X_σ. In particular $C(S_n(f)|_{X_\sigma}) \cap K$ is a finite set. Since $C_*(S_n(f)) \cap K$ is contained in $C(S_n(f)|_{X_\sigma}) \cap K$, $C_*(S_n(f)) \cap K$ is also finite. We prove the lemma by the induction on $l := \sharp(C_*(S_n(f)) \cap K)$.

The case $l = 0$ is trivial. Suppose $C_*(S_n(f)) \cap K = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_l\}$. There are open subsets $V_1, \ldots, V_l$ of X_σ such that $\boldsymbol{p}_i \in V_i$, $C(S_n(f)|_{X_\sigma}) \cap \overline{V}_i = \{\boldsymbol{p}_i\}$ $(i = 1, \ldots, l)$ and $\overline{V}_i \cap \overline{V}_j = \emptyset$ $(i \neq j)$. Since non-degenerate critical points are persistent, there is a neighborhood $\mathcal{U}$ of f in $\mathcal{C}^\infty(X \times X)$ such that for all $g \in \mathcal{U}$

(i) $C_*(S_n(g)) \cap K \subset \bigcup_{i=1}^l V_i$,

(ii) $\sharp(C(S_n(g)|_{X_\sigma}) \cap V_i) = 1$ for $i = 1, \ldots, l$. (Then, $\sharp(C(S_n(g)) \cap V_i) \leq 1$.)

Take $c > 0$ such that $f + cg_{\boldsymbol{p}_1} \in \mathcal{U}$ ($g_{\boldsymbol{p}_1}$ is the function given in Lemma 2.7.) and that $\boldsymbol{p}_1$ is a non-degenerate critical point of the following function $X^{n+1} \to \mathbf{R}$:

$$\boldsymbol{x} \mapsto S_n(f)(\boldsymbol{x}) + c\left(\sum_{j=0}^n \mu(j)|x_j - p_{1,j}|^2\right),$$

where $\boldsymbol{p}_1 = (p_{1,0}, p_{1,1}, \ldots, p_{1,n})$. (The latter condition is satisfied for all but finitely many $c \in \mathbf{R}$ by Lemma 2.4.) Put $g_1 := f + cg_{\boldsymbol{p}_1}$. From Lemma 2.7, for $\boldsymbol{x} = (x_0, \ldots, x_n) \in U_{\boldsymbol{p}_1}$,

$$S_n(g_1)(\boldsymbol{x}) = S_n(f)(\boldsymbol{x}) + c\left(\sum_{j=0}^n \mu(j)|x_j - p_{1,j}|^2\right).$$

By the choice of $g_1 \in \mathcal{U}$, $\boldsymbol{p}_1$ is the unique critical point of $S_n(g_1)$ in V_1 (see the above condition (ii)), and it is non-degenerate. Therefore we have $C_*(S_n(g_1)) \cap K \subset \bigcup_{i=2}^l V_i$. This implies $\sharp(C_*(S_n(g_1)) \cap K) \leq l - 1$. By the assumption of induction, g_1 can be approximated by $g \in \mathcal{C}^\infty(X \times X)$ such that $C_*(S_n(g)) \cap K = \emptyset$. If f is symmetric, then g_1 is also symmetric and we can choose a symmetric approximation g. $\square$

Proposition 2.9. *Let $\sigma \in \mathbf{P}_n$ and $f \in \mathcal{C}^\infty(X \times X)$. Suppose $C_*(S_n(f)) \cap \Sigma_\sigma = \emptyset$. Then f can be approximated arbitrarily well by $f' \in \mathcal{C}^\infty(X \times X)$ such that $C_*(S_n(f')) \cap X_\sigma = \emptyset$. If f is symmetric, then we can choose a symmetric approximation f'.*

Proof. Since Σ_σ is compact and $C_*(S_n(f)) \cap \Sigma_\sigma = \emptyset$, there is an open set $W_0 \subset X_\sigma$ such that $\Sigma_\sigma \subset W_0$ and $\overline{W}_0 \cap C_*(S_n(f)) = \emptyset$. Take $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k \in X_\sigma \setminus \Sigma_\sigma$ and open sets $V_1, \ldots, V_k \subset X_\sigma$ such that $\boldsymbol{p}_i \in V_i$, $\overline{V}_i \subset U_{\boldsymbol{p}_i}$ and $X_\sigma = W_0 \cup \bigcup_{i=1}^k V_i$. Put $f_0 := f$ and $W_i := W_0 \cup \bigcup_{j=1}^i V_j$ for $i = 1, \ldots, k$.

We will inductively show that if $f_i \in \mathcal{C}^\infty(X \times X)$ satisfies $C_*(S_n(f_i)) \cap \overline{W}_i = \emptyset$ then f_i can be approximated by $f_{i+1} \in \mathcal{C}^\infty(X \times X)$ satisfying $C_*(S_n(f_{i+1})) \cap \overline{W}_{i+1} = \emptyset$. (If f_i is symmetric, then we can choose f_{i+1} symmetric.) Since $C_*(S_n(f_0)) \cap \overline{W}_0 = \emptyset$ and $X_\sigma = W_0 \cup \bigcup_{i=1}^k V_i$, this will complete the proof.

Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$ be the standard basis of $\mathbf{R}^N_\sigma$ ($m = N|\sigma|$). By Proposition 2.3, for a.e. $(c_1, \ldots, c_m) \in \mathbf{R}^m$, the following is a Morse function.

$$(4)\quad \mathbf{R}^N_\sigma \supset X_\sigma \ni \boldsymbol{x} \mapsto S_n(f_i)(\boldsymbol{x}) + \left\langle \sum_{j=1}^m c_j \boldsymbol{e}_j, \boldsymbol{x} - \boldsymbol{p}_{i+1} \right\rangle \in \mathbf{R}.$$

Take small $(c_1, \ldots, c_m) \in \mathbf{R}^m$ such that (4) is a Morse function. Put $g_i := f_i + \sum_{j=1}^m c_j f_{\boldsymbol{p}_{i+1}, \boldsymbol{e}_j}$. ($f_{\boldsymbol{p}_{i+1}, \boldsymbol{e}_j}$ is the function introduced in Lemma 2.6.) Then $S_n(g_i)(\boldsymbol{x}) = S_n(f_i)(\boldsymbol{x}) + \langle \sum_{j=1}^m c_j \boldsymbol{e}_j, \boldsymbol{x} - \boldsymbol{p}_{i+1} \rangle$ for $\boldsymbol{x} \in U_{\boldsymbol{p}_{i+1}} \cap X_\sigma$. This implies $C_*(S_n(g_i)|_{X_\sigma}) \cap \overline{V}_{i+1} = \emptyset$. By Lemma 2.8, g_i can be approximated by f_{i+1} satisfying $C_*(S_n(f_{i+1})) \cap \overline{V}_{i+1} = \emptyset$.

Since $C_*(S_n(f_i)) \cap \overline{W}_i = \emptyset$ by the assumption, if we choose $(c_1, \ldots, c_m)$ sufficiently small and f_{i+1} sufficiently close to g_i then $C_*(S_n(f_{i+1})) \cap \overline{W}_i = \emptyset$. Thus we have $C_*(S_n(f_{i+1})) \cap \overline{W}_{i+1} = \emptyset$. $\square$

Proof of Theorem 2.2. Set $f_0 := f$. We will inductively construct f_i below. Let $\mathbf{P}_n = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ ($m = |\mathbf{P}_n|$), and we can assume that these are indexed so that $\sigma_i \geq \sigma_j \Rightarrow i \leq j$. If $f_i \in \mathcal{C}^\infty(X \times X)$ satisfies $C_*(S_n(f_i)) \cap (\bigcup_{j \leq i} X_{\sigma_j}) = \emptyset$, then by Proposition 2.9, f_i can be approximated by $f_{i+1} \in \mathcal{C}^\infty(X \times X)$ satisfying $C_*(S_n(f_{i+1})) \cap X_{\sigma_{i+1}} = \emptyset$. We can choose f_{i+1} sufficiently close to f_i so that $C_*(S_n(f_{i+1})) \cap (\bigcup_{j \leq i} X_{\sigma_j}) = \emptyset$. Hence $C_*(S_n(f_{i+1})) \cap (\bigcup_{j \leq i+1} X_{\sigma_j}) = \emptyset$. By induction $f = f_0$ can be approximated by $f_m \in \mathcal{C}^\infty(X \times X)$ satisfying $C_*(S_n(f_m)) = \emptyset$. If f is symmetric, then we can choose all f_i symmetric. $\square$

Appendix A. Review of Betti-number entropy. In this appendix we review the definition of Betti-number entropy introduced by Bertelson-Gromov [2]. All results in this appendix are contained in [2].

Let M be a compact connected smooth manifold. If M is oriented, then we use cohomology over $\mathbf{R}$. If M is unoriented, then we use cohomology over $\mathbf{Z}/2\mathbf{Z}$. Let $a \in H^*(M) := \bigoplus_{k \geq 0} H^k(M)$, and let $U \subset M$ be an open subset. We write $\operatorname{supp} a \subset U$ if there exists an open subset $V \subset M$ such that $M = U \cup V$ and $a|_V = 0$ in $H^*(V)$ [2, Notation 4.1].

Let X be a compact connected smooth manifold of dimension ≥ 1. Let $f : X \times X \to \mathbf{R}$ be a smooth function. For $n \geq 1$ we define $S_n(f) : X^{n+1} \to \mathbf{R}$ as in (1), and set $f_n := S_n(f)/n$. Let $\pi_n : X^{n+1} \to X^n$, $(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_{n-1})$. For an open set $U \subset X^{n+1}$ we define a subspace $A_n(U) \subset H^*(X^{n+1})$ as the set of $a \in \pi_n^*(H^*(X^n))$ satisfying $\operatorname{supp} a \subset U$. For $c \in \mathbf{R}$ and $\delta > 0$, consider the following linear map:

$$\begin{aligned} A_n(f_n^{-1}(-\infty, c+\delta)) \to \operatorname{Hom}(A_n(f_n^{-1}(c-\delta, +\infty)),\\ A_n(f_n^{-1}(c-\delta, c+\delta))),\\ a \mapsto (b \mapsto a \cup b). \end{aligned}$$

We define $b_n(c,\delta)$ as the rank of this linear map.

Lemma A.1 ([2], Lemma 5.1). *For $c, c' \in \mathbf{R}$ and $\delta > 0$,*

$$b_{n+m}(\alpha c + (1-\alpha)c', \delta) \geq b_n(c,\delta) b_m(c',\delta)$$

for $\alpha = n/(n+m)$.

Then we can define the Betti-number entropy $b(c)$ by

$$b(c) := \lim_{\delta \to 0} \left(\lim_{n \to \infty} \frac{\log b_n(c,\delta)}{n} \right).$$

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