

On discontinuous subgroups acting on solvable homogeneous spaces

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Abstract: We present in this note an analogue of the Selberg-Weil-Kobayashi local rigidity Theorem in the setting of exponential Lie groups and substantiate two related conjectures. We also introduce the notion of stable discrete subgroups of a Lie group G following the stability of an infinitesimal deformation introduced by T. Kobayashi and S. Nasrin (cf. [11]). For Heisenberg groups, stable discrete subgroups are either non-abelian or abelian and maximal. When G is threadlike nilpotent, non-abelian discrete subgroups are stable. One major aftermath of the notion of stability as reveal some studied cases, is that the related deformation spaces are Hausdorff spaces and in most of the cases endowed with smooth manifold structures.

Key words: Solvable Lie subgroup; proper action; deformation space; discontinuous subgroup; rigidity; stability.

1. Introduction. This note deals with some topological features of the deformation space of properly discontinuous actions of a discrete subgroup on a homogeneous space, namely the stability and the rigidity. The problem of describing deformations was first advocated by T. Kobayashi for the general non-Riemannian setting in [6] where he formalized the study of the deformation of Clifford-Klein forms from a theoretic point of view, (see [10] for further perspectives and basic examples). Since [11], the deformation and the moduli spaces for abelian discontinuous subgroups have been recently found explicitly in a number of settings where the underlying group G is exponential solvable (cf. [1,2] and [15]).

A classical local rigidity Theorem proved by A. Selberg and A. Weil for Riemannian symmetric spaces and generalized later by T. Kobayashi for non-Riemannian homogeneous spaces asserts that there are no continuous deformations of cocompact discontinuous groups for G/H for a linear non-compact semi-simple Lie group G except for few cases: G is not locally isomorphic to $SL_2(\mathbf{R})$ for H compact or G' is not locally isomorphic to $SO(n, 1)$

or $SU(n, 1)$ for $G = G' \times G'$ and $H = \Delta_{G'}$ (cf. [6] and [10]).

When G is nilpotent connected and simply connected, I believe that the local rigidity property does not hold (cf. Conjecture 3.1). When more generally G is assumed to be exponential solvable and $H \subset G$ a maximal subgroup, we present an analogue of the Selberg-Weil-Kobayashi Theorem stating that the local rigidity property holds if and only if the group is isomorphic to the group $ax + b$ of affine transformations of the line (cf. Theorem 3.4). In this context, we do also remove the assumption on Γ to be uniform in G/H . Our proof relies on the fact that any discontinuous group has a unique syndetic hull in the specific setting of Theorem 3.4. However, the general case remains open (see Conjecture 3.2).

We also focus on the notion of stable discontinuous subgroups following Kobayashi-Nasrin (cf. [11]), for which any associated parameter space is open independently from the nature of the subgroup H . We characterize stable discrete subgroups for Heisenberg groups and treat the case of threadlike Lie groups. Clearly, the notion of stability has important impact on the local (and global) geometric features of the related deformations. We show in many cases that stable discontinuous

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subgroups have Hausdorff deformation spaces, which are even more endowed with a manifold structure in most of the cases (Theorems 4.3 and 4.4). Detailed proofs of our results will be published elsewhere.

2. Preliminaries and notation. We begin this section with fixing some notation and terminology and recording some basic facts about deformations. The readers could consult the references [1,5,6,8–10] and some references therein for broader information about the subject. Concerning the entire subject, we strongly recommend the papers [5] and [10].

2.1. Proper and fixed point actions. Let \mathcal{M} be a locally compact space and K a locally compact topological group. The continuous action of the group K on \mathcal{M} is said to be:

(1) Proper if, for each compact subset $S \subset \mathcal{M}$ the set $K_S = \{k \in K : k \cdot S \cap S \neq \emptyset\}$ is compact.

(2) Fixed point free (or free) if, for each $m \in \mathcal{M}$, the isotropy group $K_m = \{k \in K : k \cdot m = m\}$ is trivial.

(3) Properly discontinuous if, K is discrete and the action of K on \mathcal{M} is proper and free.

In the case where $\mathcal{M} = G/H$ is a homogeneous space and K a subgroup of G , then it is well known that the action of K on \mathcal{M} is proper if $SHS^{-1} \cap K$ is compact for any compact set S in G . Likewise the action of K on \mathcal{M} is free if for every $g \in G$, $K \cap gHg^{-1} = \{e\}$. In this context, the subgroup K is said to be a discontinuous group for the homogeneous space \mathcal{M} , if K is a discrete subgroup of G and K acts properly and freely on \mathcal{M} .

2.2. Clifford-Klein forms. For any given discontinuous subgroup Γ for the homogeneous space G/H , the quotient space $\Gamma \backslash G/H$ is said to be a *Clifford-Klein form* for the homogeneous space G/H . The following point was emphasized in [7]. Any Clifford-Klein form is endowed with a smooth manifold structure for which the quotient canonical surjection $\pi : G/H \rightarrow \Gamma \backslash G/H$ turns out to be an open covering and particularly a local diffeomorphism. On the other hand, any Clifford-Klein form $\Gamma \backslash G/H$ inherits any G -invariant geometric structure (e.g. complex structure, pseudo-Riemannian structure, conformal structure, symplectic structure,...) on the homogeneous space G/H through the covering map π .

2.3. Parameter and deformation spaces. The material dealt with in this subsection comes

from [10]. The reader could also consult the references [6] and [9] for precise definitions. As in the first introductory section, we designate by $\text{Hom}(\Gamma, G)$ the set of group homomorphisms from Γ to G endowed with the point-wise convergence topology. The same topology is obtained by taking generators $\gamma_1, \dots, \gamma_k$ of Γ , then using the injective map

$$\text{Hom}(\Gamma, G) \rightarrow G \times \dots \times G, \varphi \mapsto (\varphi(\gamma_1), \dots, \varphi(\gamma_k))$$

to equip $\text{Hom}(\Gamma, G)$ with the relative topology induced from the direct product $G \times \dots \times G$. The related parameter space $\mathcal{R}(\Gamma, G, H)$ which consists of all $\varphi \in \text{Hom}(\Gamma, G)$ for which φ is injective and $\varphi(\Gamma)$ acts properly discontinuously on G/H is introduced by T. Kobayashi in [6] for general settings, and stands for an interesting object of the set $\text{Hom}(\Gamma, G)$. Such a space plays a crucial role as we will see later.

For $\varphi \in \mathcal{R}(\Gamma, G, H)$, the space $\varphi(\Gamma) \backslash G/H$ is a Clifford-Klein form which is a Hausdorff topological space and even equipped with a structure of a smooth manifold for which, the quotient canonical map is an open covering. Let now $\varphi \in \mathcal{R}(\Gamma, G, H)$ and $g \in G$, we consider the element φ^g of the set $\text{Hom}(\Gamma, G)$ defined by $\varphi^g(\gamma) = g\varphi(\gamma)g^{-1}, \gamma \in \Gamma$. It is then clear that the element $\varphi^g \in \mathcal{R}(\Gamma, G, H)$ and that the map:

$$\begin{aligned} \varphi(\Gamma) \backslash G/H &\longrightarrow \varphi^g(\Gamma) \backslash G/H, \\ \varphi(\Gamma)xH &\mapsto \varphi^g(\Gamma)g^{-1}xH \end{aligned}$$

is a natural diffeomorphism. Following [10], we consider then the orbit space

$$\mathcal{T}(\Gamma, G, H) = \mathcal{R}(\Gamma, G, H)/G$$

instead of $\mathcal{R}(\Gamma, G, H)$ in order to avoid the unessential part of deformations arising inner automorphisms and to be more precise on parameters. We call the set $\mathcal{T}(\Gamma, G, H)$ the deformation space of the action of Γ on the homogeneous space G/H .

3. The Selberg-Weil-Kobayashi rigidity Theorem.

3.1. The terminology of rigidity. We keep the same notation. A. Weil [14] introduced the notion of local rigidity of homomorphisms in the case where the subgroup H is compact. T. Kobayashi [6] generalized it in the case where H is not compact. For non-Riemannian setting G/H with H non-compact, the local rigidity does not hold in general. In the reductive case, T. Kobayashi

first proved in [4] that local rigidity may fail even for irreducible symmetric spaces of high dimensions. For non-compact setting, the local rigidity does not hold in general in the non-Riemannian case studied in [1,2,5,10,11]. We briefly recall here some details. For a comprehensible information, we refer the readers to the references [1,3–11]. For $\varphi \in \mathcal{R}(\Gamma, G, H)$, the discontinuous subgroup $\varphi(\Gamma)$ for the homogeneous space G/H is said to be *locally rigid* (resp. *rigid*) (cf. [6]) as a discontinuous group of G/H if the orbit of φ through the inner conjugation is open in $\mathcal{R}(\Gamma, G, H)$ (resp. in $\text{Hom}(\Gamma, G)$). This means equivalently that any point sufficiently close to φ should be conjugate to φ under an inner automorphism of G . So, the homomorphisms which are locally rigid are those which are isolated points in the deformation space $\mathcal{T}(\Gamma, G, H)$. When every point in $\mathcal{R}(\Gamma, G, H)$ is locally rigid, the deformation space turns out to be discrete. If a given $\varphi \in \mathcal{R}(\Gamma, G, H)$ is not locally rigid, we say that it admits a *continuous deformation*.

Once specified the setting where G is a nilpotent connected and simply connected, I believe that the following conjecture holds:

Conjecture 3.1. *Let G be a connected simply connected nilpotent Lie group, H a connected subgroup of G and Γ a non-trivial discontinuous subgroup for G/H . Then, the local rigidity property fails to hold.*

Remark 3.1. Provided that Γ is abelian, Conjecture 3.1 holds. Indeed, let $\mathfrak{l}, \mathfrak{h}$ and \mathfrak{g} designate the Lie algebras of the syndetic hull of Γ , H and G . Then $\mathcal{R}(\Gamma, G, H)$ is homeomorphic to

$$\mathcal{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) := \{ \psi \in \text{Hom}(\mathfrak{l}, \mathfrak{g}) : \dim \psi(\mathfrak{l}) = \dim \mathfrak{l} \text{ and } \exp(\psi(\mathfrak{l})) \text{ acts properly on } G/H \}$$

as originated in [11] and proved in [1]. Let $\varphi \in \mathcal{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ and $L_\varphi = \mathbf{R}_{>0} \cdot \varphi$. Clearly, $L_\varphi \subset \mathcal{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ and $G \cdot \psi \cap L_\varphi = \{ \psi \}$ for any $\psi \in L_\varphi$, as G acts unipotently on $\text{Hom}(\mathfrak{l}, \mathfrak{g})$. The projection

$$\pi : L_\varphi \rightarrow \mathbf{R}_{>0} \cdot [G \cdot \varphi] \subset \mathcal{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$$

is bijective, continuous and open and therefore bicontinuous. This means that $G \cdot \varphi$ can not be open in $\mathbf{R}_{>0} \cdot [G \cdot \varphi]$.

We now pay attention to the general context where G is connected and solvable.

3.2. The Selberg-Weil-Kobayashi rigidity Theorem. We first record the following the result:

Theorem 3.2 (Local rigidity Theorem: Selberg and Weil [14]). *Let G be a non-compact linear simple Lie group and H its maximal compact subgroup, then the following assertions are equivalent:*

- (1) *There exists a uniform lattice $\varphi : \Gamma \rightarrow G$ such that $\varphi \in \mathcal{R}(\Gamma, G, H)$ admits continuous deformations.*
- (2) *G is locally isomorphic to $SL_2(\mathbf{R})$.*

For broader information, the author could also consult the references [12] and [13]. A parallel upshot to encompass the non-Riemannian setup is as follows:

Theorem 3.3 (Local rigidity Theorem: T. Kobayashi [9]). *We keep the same assumptions as in Theorem 3.2 and let $(G', H') := (G \times G, \Delta_G)$, where Δ_G denotes the diagonal group. Then the following assertions are equivalent:*

- (1) *There exists a uniform lattice $\varphi : \Gamma \rightarrow G$ such that $\varphi \times 1 \in \mathcal{R}(\Gamma, G', H')$ admits continuous deformations.*
- (2) *G is locally isomorphic to $SO(n, 1)$ or $SU(n, 1)$.*
- (3) *G does not have Kazhdan's property (T).*

Note that Theorem 3.2 was formulated so that these two rigidity Theorems can be compared. As such, the result of Theorem 3.3 produces some irreducible non-Riemannian symmetric spaces of arbitrarily high dimension endowed with a uniform lattice for which the local rigidity does not hold. For the Riemannian case, this is very rare.

We are concerned with an analogue of Theorems 3.2 and Theorem 3.3 in the context of a real solvable Lie group G .

Let $\mathfrak{g} = \mathbf{R}\text{-span}(X, Y)$ be the Lie algebra of the $ax + b$ group with the Lie bracket $[X, Y] = Y$. For $\mathfrak{h} = \mathbf{R}X$ and Γ any discontinuous subgroup for $\exp(\mathfrak{g})/\exp(\mathfrak{h})$, the local rigidity property holds. Indeed, if Γ is non-trivial, it is isomorphic to $\exp(\mathbf{Z}Y)$. The corresponding parameter space is then homeomorphic to $\mathbf{R}^\times Y$. For $\varphi = aY \in \mathcal{R}(\Gamma, G, H)$ with $a \in \mathbf{R}^\times$, we have

$$G \cdot \varphi = \{ ae^b Y, b \in \mathbf{R} \}.$$

This means that $\mathcal{R}(\Gamma, G, H)$ only admits two open orbits.

For the general exponential case, we prove the following:

Theorem 3.4 (The analogue of Selberg-Weil-Kobayashi rigidity Theorem). *Let G be an exponential Lie group, H a non-normal connected*

maximal subgroup of G and Γ a discontinuous subgroup for G/H . Then, the following conditions are equivalent:

- i) G is isomorphic to the $ax + b$ group.
- ii) Every homomorphism in $\mathcal{R}(\Gamma, G, H)$ is locally rigid.
- iii) Some homomorphism in $\mathcal{R}(\Gamma, G, H)$ is locally rigid.

One major ingredient to prove this result, is that any discrete subgroup acting freely on a maximal homogeneous space turns out to be abelian and therefore admits a syndetic hull as G is exponential. When we look at the general solvable setting, abelian discrete subgroups may fail to admit such a syndetic hull as the exponential map is no longer a diffeomorphism. In these circumstances, I believe that the following conjecture holds:

Conjecture 3.2. *Let G be a connected simply connected solvable Lie group, H a maximal non-normal connected subgroup of G and Γ a non-trivial discontinuous subgroup for G/H . Then, the local rigidity property holds if and only if G is isomorphic to the $ax + b$ group.*

4. On stable discrete subgroups.

4.1. The terminology of stability in the sense of Kobayashi-Nasrin. Let (G, H, Γ) as in the most general setting (G is a Lie group, H a connected closed subgroup of G and Γ a discontinuous subgroup for the homogeneous space G/H). The homomorphism $\varphi \in \mathcal{R}(\Gamma, G, H)$ is said to be *stable* in the sense of Kobayashi-Nasrin [11], if there is an open set in $\text{Hom}(\Gamma, G)$ which contains φ and is contained in $\mathcal{R}(\Gamma, G, H)$. When the set $\mathcal{R}(\Gamma, G, H)$ is an open subset of $\text{Hom}(\Gamma, G)$, then obviously each of its elements is stable which is the case for any irreducible Riemannian symmetric spaces with the assumption that Γ is torsion free uniform lattice of G ([11] and [14]).

Furthermore, we point out in this setting that the concept of stability may be one fundamental genesis to understand the local structure of the deformation space.

Remark 4.1. It is straightforward to see that a point in $\mathcal{R}(\Gamma, G, H)$ is rigid if and only if it is locally rigid and stable.

4.2. Stability of discrete subgroups. Let G be a locally compact group and Γ a discrete subgroup of G . In ([8], (5.2.1)), T. Kobayashi defines the set $\mathfrak{h}(\Gamma : G)$ consisting of subsets H

for which $SHS^{-1} \cap \Gamma$ is compact for any compact set S in G . Let $\mathfrak{h}_{gp}(\Gamma : G)$ be the set of all closed connected subgroups belonging to $\mathfrak{h}(\Gamma : G)$.

I set:

Question 4.1. *For a given discrete subgroup Γ of G , is it possible to characterize all the subgroups $H \in \mathfrak{h}_{gp}(\Gamma : G)$ for which the parameter space $\mathcal{R}(\Gamma, G, H)$ is open? (Any deformation parameter is stable in the sense of Kobayashi-Nasrin).*

An answer to this question is already given for some restrictive cases of nilpotent Lie groups. This question naturally leads to the following notion of stability:

Definition 4.2. 1. Let Γ be a discrete subgroup of G . We set $\text{Stab}(\Gamma : G)$ the set of all subgroups $H \in \mathfrak{h}_{gp}(\Gamma : G)$ for which the parameter space $\mathcal{R}(\Gamma, G, H)$ is open.

2. A discrete subgroup of G is said to be stable, if $\text{Stab}(\Gamma : G) = \mathfrak{h}_{gp}(\Gamma : G)$.

I pose therefore the following question:

Question 4.2. *Is it possible to characterize all stable discrete subgroups of a connected simply connected nilpotent Lie groups?*

Let \mathfrak{g} designate the Heisenberg Lie algebra of dimension $2n + 1$ and $G = \exp(\mathfrak{g})$ the corresponding Lie group. \mathfrak{g} can be defined as a real vector space endowed with a skew-symmetric bilinear form b of rank $2n$ and a fixed generator Z belonging to the kernel of b . The center $\mathcal{Z} = \mathbf{R}Z$ of \mathfrak{g} is then the kernel of b and it is the one dimensional subspace $[\mathfrak{g}, \mathfrak{g}]$. For any $X, Y \in \mathfrak{g}$, the Lie bracket is given by $[X, Y] = b(X, Y)Z$. We have the following results:

Theorem 4.3. *Let G be a Heisenberg group and Γ a discrete subgroup of G . Then*

- 1. Γ is stable if and only if Γ is non-abelian or Γ is abelian and maximal (of rank $n + 1$).
- 2. If not, then $\text{Stab}(\Gamma : G) = \{H \in \mathfrak{h}_{gp}(\Gamma : G) : \exp(\mathcal{Z}) \subset H\}$.
- 3. For any $H \in \text{Stab}(\Gamma : G)$, the deformation space $\mathcal{S}(\Gamma, G, H)$ is equipped with a smooth manifold structure.

Here and in the rest of the paper, the rank of a discrete subgroup simply means the dimension of its syndetic hull.

We now pay attention to the setting of n -step threadlike nilpotent Lie groups. A threadlike Lie algebra \mathfrak{g} is a real n -step nilpotent Lie algebra of dimension $n + 1$ admitting a stratified basis $\{X, Y_1, \dots, Y_n\}$ with non-trivial Lie brackets:

$$[X, Y_i] = Y_{i+1}, \quad i \in \{1, \dots, n-1\}.$$

The connected simply connected associated Lie group $G = \exp(\mathfrak{g})$, is also called a n -step threadlike Lie group. In the case where $n = 2$, G coincides with the three dimensional Heisenberg Lie group. Our upshot concerning this class of nilpotent Lie groups announces as follows:

Theorem 4.4. *Let G be a threadlike Lie group. Then any non-abelian discrete subgroup Γ of G is stable. In this case, for any $H \in \mathfrak{H}_{gp}(\Gamma : G)$, we have:*

1. *The deformation space $\mathcal{T}(\Gamma, G, H)$ is a Hausdorff space.*

2. *For $k > 3$, $\mathcal{T}(\Gamma, G, H)$ is endowed with a manifold structure.*

3. *For $k = 3$, the parameter space $\mathcal{R}(\Gamma, G, H)$ is a disjoint union of an open dense smooth manifold of dimension $n + 4$ and a closed smooth manifold of dimension $n + 3$. Accordingly, $\mathcal{T}(\Gamma, G, H)$ is a disjoint union of an open dense smooth manifold and a closed smooth manifold.*

Let now G designate an exponential Lie group, H a connected maximal subgroup of G and Γ a discrete subgroup of G acting on G/H as a discontinuous group. If H is normal, then H is of codimension one and the related parameter space is open. Otherwise, we have the following theorem:

Theorem 4.5. *Let G be an exponential Lie group, H a non-normal connected maximal subgroup of G and Γ a discontinuous subgroup for G/H . Then the parameter space $\mathcal{R}(\Gamma, G, H)$ is open if and only if Γ is of rank two.*

I close the paper by asking the following question:

Question 4.3. *Let G be a connected simply connected nilpotent Lie group and Γ a stable subgroup of G as in definition (4.2, 2). Is it true that for any $H \in \mathfrak{H}_{gp}(\Gamma : G)$, the deformation space $\mathcal{T}(\Gamma, G, H)$ is a Hausdorff space?*

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