On Yoshida's conjecture on the derivative of Shintani zeta functions

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Abstract: The purpose of this paper is to prove a conjecture in Yoshida's book [2, p.33] on the higher derivative of Shintani zeta functions at $s = 0$. We use multivariable Shintani zeta functions to prove the conjecture.

Key words: Shintani's formula; multiple zeta function; Shintani zeta function.

1. Introduction. Let $A = (a_{ij})$, $1 \leq i \leq n$, $1 \leq j \leq r$ be an $n \times r$-matrix. For $1 \leq i \leq n$, let $A^{(i)}$ denote the i-th row of A. We assume $a_{ij} > 0$ for all i and j. Let b be a column vector such that $b = {}^t(b_1, b_2, \dots, b_n)$, $b_i > 0$ for all i. We define the function $\zeta_0(\boldsymbol{s}, A, b)$ for $\boldsymbol{s} = (s_1, \dots, s_n)$ by

$$\zeta_0(\boldsymbol{s}, A, b) = \sum_{m_1, \dots, m_r \geq 0} \prod_{i=1}^{n} (b_i + a_{i1} m_1 + \dots + a_{ir} m_r)^{-s_i}.$$

This definition is due to Hida's book [1, p.48]. This series converges absolutely and locally uniformly if $Re(s_1 + \dots + s_n) > r$ (Lemma 2). It is known that $\zeta_0(\boldsymbol{s}, A, b)$ can be continued meromorphically to the whole $\mathbf{C}^n$ and we denote this (meromorphically continued) function by $\zeta(\boldsymbol{s}, A, b)$. $\zeta_0((s, \dots, s), A, b)$ can be continued meromorphically to the whole $\mathbf{C}$ (Lemma 5) and we denote it by $Z(s, A, b)$. $Z(s, A, b)$ is holomorphic at $s = 0$. Shintani defined the one variable zeta functions $Z(s, A, b)$ and expressed a Hecke L-function of a totally real field by sum of $Z(s, A, b)$ using cone decomposition. Moreover, he expressed the first derivative at $s = 0$ of $Z(s, A, b)$ by sum of first derivative at $s = 0$ of $Z(s, A^{(i)}, (b_i))$ and an elementary term (see [3–5]). There is a similar formula which express the first derivative at $s = 0$ of $Z(s, A, b)$ by sum of $Z(s, A^{(i)}, (b_i))$ and $Z\left(s, \begin{pmatrix} A^{(i_1)} \\ A^{(i_2)} \end{pmatrix}, \begin{pmatrix} b_{i_1} \\ b_{i_2} \end{pmatrix}\right)$.

$$\begin{aligned}(1)\quad &\left(\frac{\partial}{\partial s}\right) Z(s, A, b)\Big|_{s=0} \\ &= - \sum_{1 \leq i \leq n} \left(\frac{\partial}{\partial s}\right) Z(s, A^{(i)}, (b_i))\Big|_{s=0} \\ &+ \frac{1}{n} \sum_{1 \leq i_1, i_2 \leq n} \left(\frac{\partial}{\partial s}\right) Z\left(s, \begin{pmatrix} A^{(i_1)} \\ A^{(i_2)} \end{pmatrix}, \begin{pmatrix} b_{i_1} \\ b_{i_2} \end{pmatrix}\right)\Bigg|_{s=0}.\end{aligned}$$

There is a second derivate version of (1).

$$\begin{aligned}(2)\quad &\left(\frac{\partial}{\partial s}\right)^2 Z(s, A, b)\Big|_{s=0} \\ &= \frac{n}{2} \sum_{1 \leq i \leq n} \left(\frac{\partial}{\partial s}\right)^2 Z(s, A^{(i)}, (b_i))\Big|_{s=0} \\ &- \sum_{1 \leq i_1, i_2 \leq n} \left(\frac{\partial}{\partial s}\right)^2 Z\left(s, \begin{pmatrix} A^{(i_1)} \\ A^{(i_2)} \end{pmatrix}, \begin{pmatrix} b_{i_1} \\ b_{i_2} \end{pmatrix}\right)\Bigg|_{s=0} \\ &+ \frac{1}{2n} \sum_{1 \leq i_1, i_2, i_3 \leq n} \left(\frac{\partial}{\partial s}\right)^2 Z\left(s, \begin{pmatrix} A^{(i_1)} \\ A^{(i_2)} \\ A^{(i_3)} \end{pmatrix}, \begin{pmatrix} b_{i_1} \\ b_{i_2} \\ b_{i_3} \end{pmatrix}\right)\Bigg|_{s=0}.\end{aligned}$$

In studying of derivative value of Artin L-function involving Hecke L-function at s = 0, Yoshida gave and proved equation (1) and (2), and conjectured more general formula (Conjecture 3.2 in [2, p.33]). The purpose of this paper is to prove the following theorem which is equivalent to the conjecture.

Theorem 1. *We have*

$$\begin{aligned}&\left(\frac{\partial}{\partial s}\right)^p Z(s, A, b)\Big|_{s=0} \\ &= \frac{1}{n} \sum_{k=0}^{p} \frac{(-n)^{p-k}}{k!(p-k)!} \sum_{1 \leq i_1, \dots, i_{k+1} \leq n} \\ &\left(\frac{\partial}{\partial s}\right)^p Z\left(s, \begin{pmatrix} A^{(i_1)} \\ \vdots \\ A^{(i_{k+1})} \end{pmatrix}, \begin{pmatrix} b_{i_1} \\ \vdots \\ b_{i_{k+1}} \end{pmatrix}\right)\Bigg|_{s=0}.\end{aligned}$$

2. Some lemmas. To prove the theorem, we will need some lemmas.

Lemma 2. *$\zeta_0(\boldsymbol{s}, A, b)$ converges absolutely and locally uniformly if $Re(s_1 + \dots + s_n) > r$.*

Proof. To prove this lemma, it is sufficient to prove that $\zeta_0(\boldsymbol{s}, A, b)$ converges if $s_1, \dots, s_n \in \mathbf{R}$ and $s_1 + \dots + s_n > r$. Since $(b_i + a_{i1} m_1 + \dots + a_{ir} m_r)/$

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$(1+r+m_1+\cdots+m_r)$ and its inverse are bounded, the convergence of $\zeta_0(\boldsymbol{s},A,b)$ is followed from the convergence of the left hand side of inequality

$$\Sigma(1+r+m_1+\cdots+m_r)^{-(s_1+\cdots+s_r)} \leq \int_0^\infty\cdots\int_0^\infty (1+m_1+\cdots+m_r)^{-(s_1+\cdots+s_r)} dm_1\cdots dm_r.$$

The convergence of the right hand side of this inequality is obvious. □

Lemma 3. *$\zeta(\boldsymbol{s},A,b)$ has the following expression*

$$\zeta(\boldsymbol{s},A,b)=\frac{f(\boldsymbol{s})}{e^{2\pi i(s_1+\cdots+s_n)}-1},$$

where $f(\boldsymbol{s})$ is a holomorphic function on $\mathbf{C}^n$.

Proof. In the case $n=1$, it follows from Theorem 1 in Section 2.4 in [1, p.53]. We assume $n\geq 2$. $\zeta(\boldsymbol{s},A,b)$ has the following expression from Theorem 1 in Section 2.4 in [1, p.53].

$$\zeta(\boldsymbol{s},A,b)=\frac{f_1(\boldsymbol{s})}{(e^{2\pi i(s_1+\cdots+s_n)}-1)\prod_{j=1}^n(e^{2\pi i s_j}-1)}$$

where $f_1(\boldsymbol{s})$ is holomorphic on the whole $\mathbf{C}^n$. $f_1(\boldsymbol{s})=0$ when $s_1+\cdots+s_n$ is large enough and $s_j-m=0$ for some $m\in\mathbf{Z}$ and some j with $1\leq j\leq n$, since $\zeta(\boldsymbol{s},A,b)$ is holomorphic when $s_1+\cdots+s_n$ is large enough. By analytic continuation, we see that (s_j-m) is not a polar divisor of $f_1(\boldsymbol{s})/(s_j-m)$. It follows that

$$f(\boldsymbol{s})=\frac{f_1(\boldsymbol{s})}{\prod_{j=1}^n(e^{2\pi i s_j}-1)}$$

is holomorphic on $\mathbf{C}^n$. □

Lemma 4. *$\zeta(\boldsymbol{s},A,b)\times(s_1+\cdots+s_n)$ is holomorphic at $\boldsymbol{s}=(0,\ldots,0)$. The constant term of Taylor expansion of $\zeta(\boldsymbol{s},A,b)\times(s_1+\cdots+s_n)$ at $\boldsymbol{s}=(0,\ldots,0)$ is 0.*

Proof. It follows from the Theorem 1 in Section 2.4 in [1, p.53]. □

Lemma 5. *If $t_1+\cdots+t_n>0$ then $\zeta_0((st_1,\ldots,st_n),A,b)$ can be continued meromorphically to the whole $\mathbf{C}$ and this analytic continuation is given by*

$$\zeta((st_1,\ldots,st_n),A,b).$$

Proof. It follows from Lemma 3. □

Lemma 6. *Let the Taylor expansion of $\zeta(\boldsymbol{s},A,b)\times(s_1+\cdots+s_n)$ at $\boldsymbol{s}=(0,\ldots,0)$ be*

$$\zeta(\boldsymbol{s},A,b)\times(s_1+\cdots+s_n)=\sum_{d=1}^\infty F_d(s_1,\ldots,s_n),$$

where $F_d(s_1,\ldots,s_n)$ is a homogeneous polynomial of $s_1,\ldots,s_n$ of degree d. Then for all $p\geq 0$ and $t_1+\cdots+t_n\neq 0$ the following identity holds.

$$\left(\frac{\partial}{\partial s}\right)^p \zeta((st_1,\ldots,st_n),A,b)\bigg|_{s=0}\times(t_1+\cdots+t_n) = p!\times F_{p+1}(t_1,\ldots,t_n)$$

Proof. It is obvious from the following expansion.

$$\zeta((st_1,\ldots,st_n),A,b)\times(t_1+\cdots+t_n) = \sum_{d=0}^\infty s^d F_{d+1}(t_1,\ldots,t_n).$$

□

Lemma 7. *Let h be any homogeneous polynomial function of degree $d\geq 1$ on a vector space V. Then the following identity holds for any $X_1,\ldots,X_n\in V$.*

$$h(X_1+\cdots+X_n) = \sum_{k=0}^d \frac{(-n)^{d-k}}{k!(d-k)!}\sum_{1\leq i_1,\ldots,i_k\leq n} h(X_{i_1}+\cdots+X_{i_k}). \tag{3}$$

Proof. First we prove that the following identity holds as polynomial in variables $Y_1,\ldots,Y_n$.

$$(Y_1+\cdots+Y_n)^d = \sum_{k=1}^d \frac{(-n)^{d-k}}{k!(d-k)!}\sum_{1\leq i_1,\ldots,i_k\leq n}(Y_{i_1}+\cdots+Y_{i_k})^d. \tag{4}$$

The right hand side of (4) equals coefficient of t^d of

$$\sum_{k=0}^d \frac{(-n)^{d-k}}{k!(d-k)!}\sum_{1\leq i_1,\ldots,i_k\leq n} e^{t(Y_{i_1}+\cdots+Y_{i_k})}\times d! \tag{5}$$

and (5) equals

$$(e^{tY_1}+\cdots+e^{tY_n}-n)^d.$$

It follows that the equation (4) holds.

Let $H(x_1,\ldots,x_d)$ be a symmetric multilinear form such that $H(x,\ldots,x)=h(x)$. We define the linear function j on the vector space consisting of homogeneous polynomial in $Y_1,\ldots,Y_n$ of degree d by

$$j(Y_{i_1}\times\cdots\times Y_{i_d})=H(X_{i_1},\ldots,X_{i_d}).$$

Now applying j to the both sides of (4), we get the (3). □

3. Proof of the theorem. Set

$$h'(\boldsymbol{t})=p!F_{p+1}(t_1,\ldots,t_n),$$

where $\boldsymbol{t}=(t_1,\ldots,t_n)\in\mathbf{C}^n$. We apply Lemma 7 for $V=\mathbf{C}^n$, $h=h'$, $X_1=(1,0,\ldots,0)$, $X_2=(0,1,$

$0,\dots,0)$, $\cdots$, $X_n=(0,\dots,0,1)$. Then by using Lemma 5 and Lemma 6, we have

$$\left(\frac{\partial}{\partial s}\right)^p Z(s,A,b)\bigg|_{s=0}\times n$$
$$=\sum_{k=1}^{p+1}\frac{(-n)^{p+1-k}}{k!(p+1-k)!}\sum_{1\le i_1,\dots,i_k\le n}$$
$$\left(\frac{\partial}{\partial s}\right)^p Z\left(s,\begin{pmatrix}A^{(i_1)}\\ \vdots\\ A^{(i_k)}\end{pmatrix},\begin{pmatrix}b_{i_1}\\ \vdots\\ b_{i_{k+1}}\end{pmatrix}\right)\bigg|_{s=0}\times k.$$

It is equivalent to the theorem.

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