A remark on parametric resonance for wave equations with a time periodic coefficient

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(Communicated by Masaki KASHIWARA, M.J.A., Sept. 12, 2011)

Abstract: The Cauchy problem for a wave equation with a time periodic coefficient is considered. We prove that if one of the initial data is a compactly supported smooth function and the other initial data is zero, then the energy of the solution of the Cauchy problem grows exponentially. This result is proved by applying the unstable properties of Hill's equation.

Key words: Energy; Hill's equation; initial value problems; resonances; wave equations.

An instability phenomenon which arises in a system when some parameter of the system varies in time is called parametric resonance. An example of parametric resonance is known as the existence of unbounded solutions of Hill's equation, see Lemma 1. We consider parametric resonance for the wave equation

$$\partial_t^2 u - a(t)\Delta u = 0 \quad \text{in} \quad \mathbf{R}^{1+n} \tag{1}$$

where $a(t) \in C^\infty(\mathbf{R})$ is 1-periodic, not a constant, positive on $\mathbf{R}$. The next is our main result.

Theorem 1. *Let $u(t,x)$ be the solution of* (1) *such that one of the initial values of u is a compactly supported smooth function, is not identically zero and the other initial value of u is identically zero and let $2 \le q \le \infty$. Then we have with positive constants δ and C*

$$\min\{\|u(m,\cdot)\|_{L^q}, \|\partial_{x_i} u(m,\cdot)\|_{L^q}, \|\partial_t u(m,\cdot)\|_{L^q}\} \ge C\exp(\delta m)$$

for sufficiently large $m \in \mathbf{N}$, where $i = 1,\dots,n$.

In this note we shall show that Theorem 1 follows from unstable properties of Hill's equation immediately. Colombini and Spagnolo [2] gave simple proof of the existence of an instability interval for Hill's equation with a coefficient in L^1_{loc} and studied homogenization for weakly hyperbolic equations with rapidly oscillating coefficients using unstable properties of Hill's equation. We refer [1] as a related result. Subsequently, Reissig and Yagdjian [6] showed that some L^p–L^q estimate for (1) is not fulfilled employing the same properties of Hill's equation, a representation formula for solutions of Hill's equation and finite propagation speed for (1), see Theorem 1 in [6]. We note that in the proof of Theorem 1 in [6] a sequence of solutions $\{u_m\}$ of (1) with smooth and compactly supported initial values is constructed such that $|u_m(m,\cdot)|$, $|\partial_{x_i} u_m(m,\cdot)|$ are uniformly greater than $C_1\exp(C_2 m)$ in bounded domains, but the support of initial values of u_m spread out as m tends to infinity. Based on a similar idea to [6] Yagdjian [7] studied nonexistence of solutions for Cauchy problems of some nonlinear wave equations with time-dependent and oscillating coefficients. Following Colombini and Rauch [4], Doi, Nishitani and Ueda [5] constructed examples of wave equations such that the coefficients are constant outside some compact spatial domain and their solution operators grow exponentially when the space dimension is greater than or equal to two. In addition, when the space dimension is equal to one, there is a study by Colombini and Rauch [3] related to [5].

Proof of Theorem 1. We shall only prove Theorem 1 in the case where $u(0,\cdot)$ is identically zero, because the other case can be shown similarly. Let $u(t,x)$ be the solution of (1) such that $u(0,\cdot)$ is identically zero and $\partial_t u(0,\cdot) \in C_0^\infty(\mathbf{R}^n)$ is not identically zero. Then $\widehat{u}(t,\xi)$ is the solution of initial value problems for the following Hill's equation with a parameter $\xi \in \mathbf{R}^n$

$$\partial_t^2 \widehat{u}(t,\xi) + |\xi|^2 a(t)\widehat{u}(t,\xi) = 0$$

with the initial values $\partial_t \widehat{u}(0,\xi)$, $\widehat{u}(0,\xi) = 0$ where $\widehat{v}(t,\xi)$ denotes the Fourier transform of $v(t,x)$ with

2010 Mathematics Subject Classification. Primary 35L15, 34C11, 35B34.

doi: 10.3792/pjaa.87.128

respect to x. Hence we consider Hill's equation with a parameter $\lambda \in \mathbf{R}$

$$v''(t) + \lambda a(t)v(t) = 0. \tag{2}$$

Let us denote by $X_\lambda(t, 0)$ the fundamental matrix for the solution operator of (2) taking ${}^t(v'(0, \lambda)\ v(0, \lambda))$ to ${}^t(v'(t, \lambda)\ v(t, \lambda))$ and denote (i, j)-element of $X_\lambda(1, 0)$ by $b_{ij}(\lambda)$, $1 \le i, j \le 2$. The following is well known, for example see Theorem 2 in [2].

Lemma 1. *There exists an open interval* $\Lambda \subset (0, \infty)$ *such that* $(\mathrm{Tr}\, X_\lambda(1, 0))^2 > 4$ *for all* $\lambda \in \Lambda$. *Moreover, for every parameter* $\lambda \in \Lambda$ *one can find smooth functions* $p(t, \lambda), q(t, \lambda)$ *and a positive number* $\gamma(\lambda)$ *so that* p *and* q *are* 1*-periodic or* 1*-semiperiodic and every solution* $v(t, \lambda)$ *of* (2) *can be written in the form*

$$v(t, \lambda) = c_1 p_1(t, \lambda) e^{\gamma(\lambda)t} + c_2 p_2(t, \lambda) e^{-\gamma(\lambda)t}$$

where c_1, c_2 *are some complex numbers.*

Owing to Lemma 1, for $\lambda \in \Lambda$ we can denote by $\mu(\lambda), \mu(\lambda)^{-1}$ the eigenvalues of $X_\lambda(1, 0)$ with $|\mu(\lambda)| > 1$. For simplicity we shall often omit to write a variable λ. By an argument similar to the proof of Lemma 2.1 in [7] it follows that there exists a closed interval $\Lambda^0 \subset \Lambda$ such that the interior of Λ^0 is not empty, $b_{12}(\lambda) \neq 0$ and $b_{21}(\lambda) \neq 0$ on Λ^0. Hence we see that $\mu \neq b_{11}$ and $\mu \neq b_{22}$ on Λ^0. Indeed, we have

$$\begin{aligned}&(\mu - b_{11})(\mu - b_{22})\\ &\quad = \mu^2 - (b_{11} + b_{22})\mu + b_{11}b_{22} = b_{12}b_{21} \neq 0\end{aligned}$$

on Λ^0. Let $\lambda \in \Lambda^0$ and $U(t, \lambda)$ be the solution of (2) with the initial values $U'(0, \lambda) = 1$, $U(0, \lambda) = 0$. The following formulas are deduced from the proof of Lemma 2.2 in [7]: For every $m \in \mathbf{N}$ we have

$$\begin{aligned} U(m) &= \frac{b_{21}}{\mu - \mu^{-1}} (\mu^m - \mu^{-m}), \\ U'(m) &= \frac{\mu - b_{22}}{\mu - \mu^{-1}} \mu^m + \frac{\mu - b_{11}}{\mu - \mu^{-1}} \mu^{-m}. \end{aligned} \tag{3}$$

We shall prove the inequality in Theorem 1 only for the L^q-norm of u, because the other inequalities in Theorem 1 can be shown in the same way. Lemma 1 and the definition of μ provide that $\min_{\Lambda^0} |b_{21}/(\mu - \mu^{-1})|$ and δ are positive where $\delta = \ln(\min_{\Lambda^0} |\mu|)$. One can find a closed domain I in $\mathbf{R}^n$ so that the set $\{|\xi|^2; \xi \in I\}$ is contained in Λ^0 and $\partial_t \widehat{u}(0, \xi) \neq 0$ in I, because $\partial_t \widehat{u}(0, \xi)$ is real analytic in $\mathbf{R}^n$ and not identically zero. Hence it follows from (3) that with a constant $C_1 > 0$

$$\begin{aligned}\inf_{\xi \in I} |\widehat{u}(m, \xi)| &= \inf_{\xi \in I} \left| U(m, |\xi|^2) \partial_t \widehat{u}(0, \xi) \right| \\ &\ge C_1 \exp(\delta m)\end{aligned}$$

for sufficiently large $m \in \mathbf{N}$. Thus we obtain the inequality for the L^2-norm of u in Theorem 1, because Parseval's equality gives

$$\int |u(m, x)|^2\, dx \ge (2\pi)^{-n} \int_I |\widehat{u}(m, \xi)|^2\, d\xi.$$

The finite propagation speed for (1) implies that for some constant C_2 the support of $u(t, \cdot)$ is contained in $\{x \in \mathbf{R}^n; |x| \le C_2(t + 1)\}$ for all $t \ge 0$. Hence Hölder's inequality shows that for some constant C_3

$$\|u(m, \cdot)\|_{L^2} \le C_3 (m + 1)^{n/(2p)} \|u(m, \cdot)\|_{L^{2r}},$$

where $1 \le r \le \infty$ and $1/p + 1/r = 1$, which yields the inequality for the L^q-norm of u in Theorem 1.

Acknowledgments. The author expresses his sincere gratitude to Profs. Mishio Kawashita and Karen Yagdjian for valuable suggestions.

References

[1] F. Colombini and S. Spagnolo, An example of a weakly hyperbolic Cauchy problem not well posed in C^∞, Acta Math. **148** (1982), 243–253.

[2] F. Colombini and S. Spagnolo, Hyperbolic equations with coefficients rapidly oscillating in time: a result of nonstability, J. Differential Equations **52** (1984), no. 1, 24–38.

[3] F. Colombini and J. Rauch, A simple example of localized parametric resonance for the wave equation, Serdica Math. J. **34** (2008), no. 1, 55–66.

[4] F. Colombini and J. Rauch, Smooth localized parametric resonance for wave equations, J. Reine Angew. Math. **616** (2008), 1–14.

[5] S. Doi, T. Nishitani and H. Ueda, Note on lower bounds of energy growth for solutions to wave equations, Osaka J. Math. (to appear).

[6] M. Reissig and K. Yagdjian, About the influence of oscillations on Strichartz-type decay estimates, Rend. Sem. Mat. Univ. Politec. Torino **58** (2000), no. 3, 375–388.

[7] K. Yagdjian, Parametric resonance and nonexistence of the global solution to nonlinear wave equations, J. Math. Anal. Appl. **260** (2001), no. 1, 251–268.