Nonreflexivity of Banach spaces of bounded harmonic functions on Riemann surfaces

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Abstract: We give a simple, short, and easy proof to the Masaoka theorem that if Dirichlet finiteness and boundedness for harmonic functions on a Riemann surface coincide with each other, then the dimension of the linear space of Dirichlet finite harmonic functions on the Riemann surface and the dimension of the linear space of bounded harmonic functions on the Riemann surface are finite and identical. The essence of our proof lies in the observation that the former of the above two Banach spaces is reflexive while the latter is not unless it is of finite dimension.

Key words: Banach space; Dirichlet integral; Green function; harmonic function; Hilbert space; reflexive.

1. Introduction. We denote by $H(R)$ the linear space of harmonic functions u on an open (i.e. noncompact) Riemann surface R. Among various linear subspaces of $H(R)$, two subspaces $HD(R)$ and $HB(R)$ are the most fundamental (cf. e.g. [1]): one is

$$HD(R) := \{u \in H(R) : D(u; R) < +\infty\}, \tag{1.1}$$

where $D(u; R)$ is the Dirichlet integral of u taken over R, i.e.

$$\begin{aligned} D(u; R) &:= \int_R du \wedge *du \\ &= \int_R |\nabla u(z)|^2 dxdy \qquad (z = x + iy), \end{aligned} \tag{1.2}$$

which plays a powerful role in analyzing the space $H(R)$ related to the Dirichlet principle such as in solving the Dirichlet problem based upon the Weyl lemma; the other is

$$HB(R) := \{u \in H(R) : \|u; R\|_\infty < +\infty\}, \tag{1.3}$$

where $\|u; R\|_\infty$ is the supremum norm of u taken over R, i.e.

$$\|u; R\|_\infty := \sup_{z \in R} |u(z)|, \tag{1.4}$$

which also plays another important role in the analysis of $H(R)$ based upon the harmonic version of the normal family argument.

In the classification theory of Riemann surfaces, it is known (cf. e.g. [16], see also [12]) that for any $n \in \mathbf{N}$, the set of positive integers, there exists a Riemann surface R such that

$$\dim HD(R) = \dim HB(R) = n, \tag{1.5}$$

where $\dim X$ is the dimension of the linear space X in the following sense, i.e. if X has a basis consisting of a finite $n \in \mathbf{N}$ number of elements in X, then $\dim X = n$, and if there is an infinite subset $Y \subset X$ whose arbitrary finite subset is always linearly independent, then $\dim X = \infty$. In general, for any Riemann surface R, (1.5) implies that (cf. e.g. [16])

$$HD(R) = HB(R). \tag{1.6}$$

Related to this fact, Masaoka [7–9] discovered the following result.

Theorem A. *The identity* (1.6) *is equivalent to the formula* (1.5) *for some* $n \in \mathbf{N}$.

As stated above, the implication from (1.5) to (1.6) is well known and actually quite easy to derive and therefore the essential part here is the implication from (1.6) to (1.5) for some $n \in \mathbf{N}$. Under (1.6), $\dim HD(R) = \dim HB(R)$ is just trivial. Really essential assertion here is thus

Assertion 1.7. *The identity* (1.6) *implies*

$$\dim HB(R) < \infty.$$

The origial proof of the above assertion by Masaoka himself relies upon the ingeneous Doob generalization [3] to the Martin boundary setting for general

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Green spaces of the classical and important result of Douglas ([4], see also [15]) which gives a concrete characterization for a L^1 function on the boundary unit circle $\partial\mathbf{D}$ of the unit disc $\mathbf{D}$ in the complex plane to be the boundary function of a function in $HD(\mathbf{D})$ so that the above original proof of Masaoka [7–9] may be said to be not too easy and simple. The present author [10] gave an alternative proof to Assertion 1.7 based upon his result on capacities on the Royden harmonic boundary (cf. [11]), which is relatively simple, easy, and elementary.

Recently we found a surprisingly simple and easy elementary ultrashort proof of Assertion 1.7 based upon only an introductory knowledge of the functional analysis of undergraduate level to present which is the *purpose* of this paper. Roughly it goes as follows. We need a little bit more care to say the following but anyway in essence the space $HD(R)$ forms a Hilbert space so that it is a reflexive Banach space. On the other hand the Banach space $HB(R)$ is seen to be isometrically isomorphic to the continuous function space $C(S)$ on a compact Hausdorff space S, i.e. $HB(R) = C(S)$ as Banach spaces, and $C(S)$ is not reflexive unless it is of finite dimension (cf. §3 below). Therefore (1.6) must imply $\dim HB(R) = \dim C(S) < \infty$. Extremely short and simple, isn't it?

2. Relative classes. Fix a relatively compact parametric disc R_0 in R and let $W := R \setminus \overline{R}_0$. Take the linear space $H(W; \partial W)$ given by

$$H(W;\partial W) := \{u \in H(W) \cap C(R) : u|\overline{R}_0 = 0\}, \tag{2.1}$$

which is referred to as the relative class to $H(R)$. Consider the directed set $\{\Omega\}$ of regular subregions Ω of R with $\Omega \supset \overline{R}_0$, which exhuasts R. For any $u \in H(R)$, let $J_\Omega u \in H(W \cap \Omega) \cap C(R)$ with $J_\Omega u = u$ on $\overline{R}_0$ and $J_\Omega u = 0$ on $R \setminus \Omega = W \setminus \Omega$. It is easily seen that $Ju := \lim_{\Omega \uparrow R} J_\Omega u$ exists locally uniformly on R. Then set

$$Iu := u - Ju.$$

The linear operator $I : H(R) \to H(W; \partial W)$ is bijective if and only if R is hyperbolic characterized by the existence of the Green function on R or equivalently by $I1 < 1$ on W, in notation $R \notin \mathcal{O}_G$, the class of parabolic Riemann surfaces characterized by the nonexistence of Green function (cf. [13], see also [14]). Furthermore, still under the assumption $R \notin \mathcal{O}_G$, the operator I preserves the linearity, order, and the supremum norm; a bit delicate is the situation concerning the Dirichlet integrals and somehow

$$D(u; R) \leqq D(Iu; R) = D(Iu; W) \leqq +\infty$$

for every $u \in H(R)$, and, $D(Iu; W)$ and $D(u; R)$ are simultaneously finite or infinite. If $R \in \mathcal{O}_G$, then $HD(R) = HB(R) = \mathbf{R}$, the real number field (cf. e.g. [16]), and avoiding this trivial case we may and will assume hereafter in this paper that $R \notin \mathcal{O}_G$. Then

$$HX(R) \cong HX(W; \partial W) \quad \text{(isomorphic as linear spaces by } I) \tag{2.2}$$

for $X = D$ and B. Thus the condition (1.6) is equivalent to

$$HD(W; \partial W) = HB(W; \partial W) \tag{2.3}$$

and

$$\dim HB(R) < \infty$$

is equivalent to

$$\dim HB(W; \partial W) < \infty.$$

We are to prove Assertion 1.7 but after the above reduction, we now see Assertion 1.7 is equivalent to the following

Assertion 2.4. *The identity* (2.3) *implies*

$$\dim HB(W; \partial W) < \infty.$$

Thus our task is to prove the above in an easy and simple manner. The reason why we replace $H(R)$ by $H(W; \partial W)$ lies in the fact that $HB(R)$ equipped with $\|\cdot\,; R\|_\infty$ is certainly a Banach space while $HD(R)$ with $D(\cdot\,; R)^{1/2}$ is not but both of $(HB(W; \partial W), \|\cdot\,; W\|_\infty)$ and $(HD(W; \partial W), D(\cdot\,; W)^{1/2})$ are Banach spaces. We can consider the mutual Dirichlet integral

$$\begin{aligned} D(u, v; W) &:= \int_W du \wedge *dv \\ &= \int_W \nabla u(z) \cdot \nabla v(z) dxdy \qquad (z = x + iy) \end{aligned} \tag{2.5}$$

of functions u and v in $HD(W; \partial W)$ taken over W, which is the inner product on $HD(W; \partial W)$ and $D(\cdot\,; W)^{1/2}$ is the norm induced by the inner product $D(\cdot, \cdot\,; W)$. Hence $(HD(W, \partial W), D(\cdot\,; W)^{1/2})$ is a Hilbert space.

In general it is often convenient to consider one more linear subspace

$$HBD(W;\partial W) := HB(W;\partial W) \cap HD(W;\partial W).$$

Using the combined norm

$$\|u;W\|_{BD} := \|u;W\|_\infty + D(u;W)^{1/2} \tag{2.6}$$

we may also give the above subspace by

$$\begin{aligned} &HBD(W;\partial W) \\ &\quad := \{u \in H(W;\partial W) : \|u;W\|_{BD} < \infty\}. \end{aligned} \tag{2.7}$$

It is also easily seen that $(HBD(W;\partial W), \|\cdot\,;W\|_{BD})$ forms a Banach space.

Let S be the Wiener harmonic boundary of R (cf. [2,16]; see also [6]), which is a compact Hausdorff space. Then the restriction $u \mapsto u|S$ gives an isometrical isomorphism of $(HB(W;\partial W), \|\cdot\,;W\|_\infty)$ onto $(C(S), \|\cdot\,;S\|_\infty)$ so that

$$(HB(W;\partial W), \|\cdot\,;W)\|_\infty) = (C(S), \|\cdot\,;S\|_\infty) \tag{2.8}$$

as Banach spaces. Recall that the dual space $C(S)^*$ of $C(S)$ is the space of signed Radon measures on S.

3. Proof of Assertion 2.4. Since we are assuming (2.3), we have

$$\begin{aligned} HD(W;\partial W) &= HB(W;\partial W) \\ &= HBD(W;\partial W) \end{aligned} \tag{3.1}$$

simply as subsets of $H(W;\partial W)$. For simplicity we write

$$X := (HD(W;\partial W), D(\cdot\,;W)^{1/2})$$

as a Hilbert space, and

$$Y := (HB(W;\partial W), \|\cdot\,;W\|_\infty)$$

and

$$Z := (HBD(W;\partial W), \|\cdot\,;W\|_{BD})$$

as Banach spaces. We denote by $T_1 : Z \to X$ and $T_2 : Z \to Y$ linear operators given by the identity mapping. Since

$$\begin{aligned} \|T_1 z\|_X &= D(z;W)^{1/2} \\ &\leqq \|z;W\|_\infty + D(z;W)^{1/2} = \|z\|_Z \end{aligned}$$

for every $z \in Z$ and thus the operator norm $\|T_1\| \leqq 1$ and similarly

$$\begin{aligned} \|T_2 z\|_Y &= \|z;W\|_\infty \\ &\leqq \|z;W\|_\infty + D(z;W)^{1/2} = \|z\|_Z \end{aligned}$$

for every $z \in Z$ and hence the operator norm $\|T_2\| \leqq 1$, we see that T_1 and T_2 are bounded linear operators. By the Banach open mapping principle (cf. e.g. [5,17]), T_1^{-1} and T_2^{-1} are also bounded. Then the linear operators $T := T_2 \circ T_1^{-1} : X \to Y$ and $T^{-1} = T_1 \circ T_2^{-1} : Y \to X$, which are just identity mappings, are also bounded. Then, on setting $K := \max(\|T\|, \|T^{-1}\|) \in [1, +\infty)$, $T : X \to Y$ is a *K-quasiisometric isomorphism*, i.e. $T : X \to Y$ is linearly isomorphic and

$$K^{-1}\|x\|_X \leqq \|Tx\|_Y \leqq K\|x\|_X \tag{3.2}$$

for every $x \in X$. Then the dual operator $T^* : Y^* \to X^*$ and the double dual operator $T^{**} : X^{**} \to Y^{**}$ of T are also K-quasiisometric isomorphisms, where X^* is the dual space of X. Moreover, $T^{**}\hat{X} = \hat{Y}$, where $\hat{X}$ ($\hat{Y}$, resp.) is the natural injection of X (Y, resp.) into X^{**} (Y^{**}, resp.) so that $\hat{X} = X$ ($\hat{Y} = Y$, resp.) (isometrically (i.e. 1-quasiisometrically) isomorphic) as Banach spaces. Since X is *reflexive* (i.e. $\hat{X} = X^{**}$) in view of the fact that X is a Hilbert space, we see that $\hat{Y} = T^{**}\hat{X} = T^{**}X^{**} = Y^{**}$ so that Y as the Banach space $(C(S), \|\cdot\,;S\|_\infty)$ (by (2.8)) is also reflexive. As a consequence of the Alaoglu theorem (cf. e.g. [5,17]), the reflexivity of $C(S)$ implies the weak compactness of the unit ball $C(S)_1$ in $C(S)$, i.e. for any directed net $(\varphi_\lambda)_{\lambda\in\Lambda} \subset C(S)_1$ there is a subnet $(\varphi_\mu)_{\mu\in M}$ (where M is a cofinal directed subset of Λ) weakly convergent to a $\varphi \in C(S)_1$ so that $\int_S \varphi_\mu d\nu \to \int_S \varphi d\nu$ for every Borel measure ν on S and in particular $(\varphi_\mu)_{\mu\in M}$ converges pointwise to φ on S, which is seen by taking as ν the Dirac measure δ_s supported by any $s \in S$.

Choose and then fix an arbitrary point $s_0 \in S$. We denote by $\mathcal{V} = \{V\}$ the totality of open neighborhoods V of s_0. We make $\mathcal{V}$ a directed set by giving the order $V_1 \leqq V_2$ by $V_1 \supset V_2$. For each $V \in \mathcal{V}$ we assign a continuous function φ_V on S such that $\varphi_V|S \setminus V = 1$, $\varphi_V(s_0) = 0$, and $0 \leqq \varphi_V \leqq 1$ on S. By the weak compactness of $C(S)_1$, the directed net $(\varphi_V)_{V\in\mathcal{V}} \subset C(S)_1$ contains a weakly convergent subnet $(\varphi_U)_{U\in\mathcal{U}}$ ($\mathcal{U} \subset \mathcal{V}$ and $\mathcal{U}$ is a base of neighborhoods of s_0) converging weakly to a $\varphi \in C(S)_1$. Since $(\varphi_U(s))_{U\in\mathcal{U}}$ converges to $\varphi(s)$ for every $s \in S$, we see that $\varphi|S \setminus \{s_0\} = 1$ and $\varphi(s_0) = 0$. This assures that s_0 is an isolated point in S so that the compact set S consists of only isolated points. Hence there is an $n \in \mathbf{N}$ such that $S = \{s_1,\ s_2, \cdots, s_n\}$. Then $C(S) = \mathbf{R}^n$ and a fortiori

$$\dim HB(W;\partial W) = \dim C(S) = \dim \mathbf{R}^n = n \in \mathbf{N}.$$

This is what we have to derive. □

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