The integral cohomology ring of E_8/T

By Masaki NAKAGAWA

Department of General Education, Kagawa National College of Technology, 355 chokushi-cho, Takamatsu, Kagawa 761-8058, Japan

(Communicated by Kenji FUKAYA, M.J.A., Feb. 12, 2010)

Abstract: We give a complete description of the integral cohomology ring of the flag manifold E_8/T, where E_8 denotes the compact exceptional Lie group of rank 8 and T its maximal torus, by the method due to Borel and Toda. This completes the computation of the integral cohomology rings of the flag manifolds for all compact connected simple Lie groups.

Key words: Cohomology; flag manifolds; Lie groups.

1. Introduction. Let G be a compact connected Lie group and T a maximal torus of G. Then the homogeneous space G/T is called a (*full* or *complete*) *flag manifold* and plays an important role in modern mathematics. In algebraic topology, the following problem is classical:

Problem 1.1. *Determine the integral cohomology ring of the flag manifold G/T for G a compact connected simple Lie group.*

The computation of the integral cohomology ring of G/T was started by Borel in 1953 [2]. Borel considered the spectral sequence for the fibration

$$G/T \xrightarrow{\iota} BT \xrightarrow{\rho} BG,$$

where BT (resp. BG) denotes the classifying space of T (resp. G), and obtained the following description of the cohomology ring of G/T; Let k be a field of characteristic p, and suppose that $p = 0$ or G has no p-torsion. The Weyl group W of G acts naturally on T, and hence, on BT and also on $H^*(BT; k)$. Then the homomorphism

$$\iota^* : H^*(BT; k) \to H^*(G/T; k)$$

induces the following isomorphism:

$$H^*(BT; k)/(H^+(BT; k)^W) \to H^*(G/T; k), \tag{1.1}$$

where $(H^+(BT; k)^W)$ denotes the ideal of $H^*(BT; k)$ generated by W-invariant polynomials of positive degrees. This is the *Borel presentation* of the cohomology ring of G/T. Borel's method is valid for the integer coefficients when G has no torsion. From this, the cases of $SU(n)$ and $Sp(n)$ follow immediately. However, when G has p-torsion, Borel's result does not hold. In 1955, Bott and Samelson developed an algorithm for computing the integral cohomology ring of G/T by means of the so-called "Bott-Samelson K-cycle"[*1)], and determined the case of the exceptional group G_2 explicitly [5, 6, Theorem III']. The above problem could be solved theoretically by their method. However, it seems difficult to apply this to other exceptional goups F_4, E_6, E_7 and E_8. In 1975, Toda gave another useful description of the integral cohomology ring of G/T from the known results on the mod p cohomology rings $H^*(G; \mathbf{Z}/p\mathbf{Z})$ of G for all primes p and the rational cohomology ring $H^*(G/T; \mathbf{Q})$ of G/T [20, Theorem 2.1, Proposition 3.2]. Based on Toda's method, the cases of $SO(2n)$, $SO(2n+1)$ were settled by Toda and Watanabe [21, Theorem 2.1, Corollary 2.2].[*2)] The cases of F_4 and E_6 were also settled in [21, Theorems A, B]. The case of E_7 was settled by the author [18, Theorem 5.9]. The only remaining case is $G = E_8$. In this article, we determine the integral cohomology ring of E_8/T explicitly along the line of Toda's method.[*3)] This completes the computation of the integral cohomology rings of the flag manifolds for all compact connected simple Lie groups.

2. Ring of invariants of $W(E_8)$. Let E_8 be the compact simply connected simple exceptional Lie group of rank 8 and T a maximal torus. Follow-

2000 Mathematics Subject Classification. Primary 57T15; Secondary 14M15.

*1) A Bott-Samelson K-cycle is refered to as a *Bott tower* in toric topology, and a B*ott-Samelson-Demazure-Hansen variety* in representation theory.

*2) The results for $SO(2n)$, $SO(2n+1)$ also appeared in [16] by computing the "cohomology ring of the root system" due to Demazure [10].

*3) Recently Duan and Zhao also computed it in the context of the *Schubert calculus* [12].

doi: 10.3792/pjaa.86.64

ing [7], we take the simple roots $\{\alpha_i\}_{1\le i\le 8}$ and denote by $\{\omega_i\}_{1\le i\le 8}$ the corresponding fundamental weights. In topology, it is customary that roots and weights are regarded as elements of $H^2(BT;\mathbf{Z})$. Let s_i $(1\le i\le 8)$ denote the simple reflection corresponding to the simple root α_i. Then the Weyl group $W(E_8)$ of E_8 is generated by simple reflections s_i $(1\le i\le 8)$. As in [19, §2], we put

$$(2.1)\qquad \begin{aligned} &t_8=\omega_8,\ t_i=s_{i+1}(t_{i+1})\ (2\le i\le 7),\\ &t_1=s_1(t_2),\ t=\omega_2.\end{aligned}$$

Then we have

$$H^*(BT;\mathbf{Z})=\mathbf{Z}[t_1,\dots,t_8,t]/(c_1-3t)$$

for $c_i=e(t_1,\dots,t_8)$, the i-th elementary symmetric polynomial in $t_1,\dots,t_8$.

According to Chevalley [8], the ring of invariants of the Weyl group $W(E_8)$ over $\mathbf{Q}$ is generated by 8 algebraically independent polynomials (*basic invariants*) of degrees $2,8,12,14,18,20,24,30$. By computing the Chern character of the adjoint representation of E_8, of dimension 248, we obtain the basic invariants I_j $(j=2,8,12,14,18,20,24,30)$ explicitly in [19, Lemma 2.3] (see also [17, 2.3]). Thus we have the following

Lemma 2.1. *The ring of invariants of the Weyl group $W(E_8)$ over $\mathbf{Q}$ is given by*

$$H^*(BT;\mathbf{Q})^{W(E_8)}=\mathbf{Q}[I_2,I_8,I_{12},I_{14},I_{18},I_{20},I_{24},I_{30}].$$

By (1.1) and Lemma 2.1, we can compute the rational cohomology ring $H^*(E_8/T;\mathbf{Q})$ of E_8/T.

3. Integral cohomology ring of E_8/T. As mentioned in the introduction, we compute $H^*(E_8/T;\mathbf{Z})$ following Toda's method. Since E_8 is simply connected, the homomorphism

$$\iota^*:H^2(BT;\mathbf{Z})\to H^2(E_8/T;\mathbf{Z})$$

is an isomorphism. Under this isomorphism, we denote the ι^*-images of t_i and t by the same symbols. Thus $H^2(E_8/T;\mathbf{Z})$ is a free $\mathbf{Z}$-module generated by t_i $(1\le i\le 8)$ and t with a relation $c_1=3t$. In [20], Toda gave the general description of the integral cohomology ring of E_8/T. In our situation, his result is stated as follows:

Proposition 3.1 [20, Proposition 3.2]. *The integral cohomology ring of E_8/T is of the form*:

$$H^*(E_8/T;\mathbf{Z})=\frac{\mathbf{Z}[t_1,\dots,t_8,t,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_9,\gamma_{10},\gamma_{15}]}{\begin{pmatrix}\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_8,\rho_9,\rho_{10},\rho_{12},\rho_{14},\rho_{15},\\ \rho_{18},\rho_{20},\rho_{24},\rho_{30}\end{pmatrix}},$$

where $t_1,\dots,t_8,t\in H^2$ are as above, and $\gamma_i\in H^{2i}$ $(i=3,4,5,6,9,10,15)$, and

$$\begin{aligned}&\rho_1=c_1-3t,\\ &\rho_i=\delta_i-2\gamma_i\ (i=3,5,9,15),\\ &\rho_i=\delta_i-3\gamma_i\ (i=4,10),\\ &\rho_6=\delta_6-5\gamma_6.\end{aligned}$$

Here δ_i $(i=3,4,5,6,9,10,15)$ are arbitrary elements of $H^*(E_8/T;\mathbf{Z})$ satisfying

$$\begin{aligned}&\delta_3\equiv Sq^2(\rho_2),\delta_5\equiv Sq^4(\delta_3),\delta_9\equiv Sq^8(\delta_5),\\ &\delta_{15}\equiv Sq^{14}(\rho_8)\mod 2,\\ &\delta_4\equiv\mathcal{P}^1(\rho_2),\delta_{10}\equiv\mathcal{P}^3(\delta_4)\mod 3,\\ &\delta_6\equiv\mathcal{P}^1(\rho_2)\mod 5.\end{aligned}$$

Other relations ρ_j $(j=2,8,12,14,18,20,24,30)$ are determined by the maximum of the integers n_j in

$$(3.1)\qquad n_j\cdot\rho_j\equiv\iota^*(I_j)\mod(\rho_i;i<j),$$

where I_j $(j=2,8,12,14,18,20,24,30)$ are the basic invariants of the Weyl group $W(E_8)$ given in Lemma 2.1.

We will carry out his program for E_8. Fortunately, in [19, Lemma 4.2], partial computation of $H^*(E_8/T;\mathbf{Z})$ has been made up to degrees 36. Thus we need only to determine the higher relations ρ_{20}, ρ_{24} and ρ_{30} explicitly. However, it seems difficult to compute them directly from the basic invariants I_{20}, I_{24} and I_{30}. We will make use of a certain subgroup of E_8. Namely, let C be the centralizer of a one dimensional torus determined by $\alpha_i=0$ $(i\ne 8)$. The local type of C can be read off from the Dynkin diagram of E_8 [3], and we have, in fact,

$$C=T^1\cdot E_7\quad\text{and}\quad T^1\cap E_7\cong\mathbf{Z}/2\mathbf{Z},$$

where T^1 denotes a certain one-dimensional torus. Consider the fibration

$$E_7/T'\cong C/T\xrightarrow{i}E_8/T\xrightarrow{p}E_8/C,$$

where T' is a maximal torus of E_7. Since $H^*(E_8/C;\mathbf{Z})$ and $H^*(E_7/T';\mathbf{Z})$ have no torsion and vanishing odd dimensional part by Bott [4, Theorem A], the Serre spectral sequence with integer coefficients for the above fibration collapses, and the following sequence

$$\begin{aligned}\mathbf{Z}\ &\to H^*(E_8/C;\mathbf{Z})\xrightarrow{p^*}H^*(E_8/T;\mathbf{Z})\\ &\xrightarrow{i^*}H^*(C/T;\mathbf{Z})\cong H^*(E_7/T';\mathbf{Z})\to\mathbf{Z}\end{aligned}$$

is co-exact. In particular, p^* is a split monomorphism so that $H^*(E_8/T;\mathbf{Z})$ contains $H^*(E_8/C;\mathbf{Z})$ as a direct summand if we identify $\operatorname{Im} p^*$ with $H^*(E_8/C;\mathbf{Z})$.

The integral cohomology ring of E_8/C is determined in [19], which we now recall:

Theorem 3.2 [19, Theorem 4.7]. *The integral cohomology ring of E_8/C is given as follows*:

$$H^*(E_8/C;\mathbf{Z}) = \mathbf{Z}[u,v,w,x]/(r_{15}, r_{20}, r_{24}, r_{30}),$$

where $\deg u = 2$, $\deg v = 12$, $\deg w = 20$, $\deg x = 30$ *and*

$$\begin{aligned}
r_{15} &= u^{15} - 2x,\\
r_{20} &= 9u^{20} + 45u^{14}v + 12u^{10}w + 60u^8v^2 + 30u^4vw\\
&\quad + 10u^2v^3 + 3w^2,\\
r_{24} &= 11u^{24} + 60u^{18}v + 21u^{14}w + 105u^{12}v^2 + 60u^8vw\\
&\quad + 60u^6v^3 + 9u^4w^2 + 30u^2v^2w + 5v^4,\\
r_{30} &= -9x^2 - 12u^9vx - 6u^5wx + 9u^{14}vw - 10u^{12}v^3\\
&\quad - 3u^{10}w^2 + 30u^8v^2w - 35u^6v^4 + 6u^4vw^2\\
&\quad - 10u^2v^3w - 4v^5 - 2w^3.
\end{aligned}$$

Remark 3.3. *The integral cohomology ring of E_8/C is also computed by Duan and Zhao* [11, Theorem 7] *in the context of the Schubert calculus. In our forthcoming paper* [14], *we will show that Theorem* 3.2 *completely coincides with the result of Duan and Zhao by means of the divided difference operators due to Bernstein-Gelfand-Gelfand* [1] *and Demazure* [10].

The relations r_{20}, r_{24} and r_{30} of $H^*(E_8/C;\mathbf{Z})$ correspond to the relations ρ_{20}, ρ_{24} and ρ_{30} of $H^*(E_8/T;\mathbf{Z})$ respectively. In order to make the description of $H^*(E_8/T;\mathbf{Z})$ complete, we have to specify the elements u, v, w and x explicitly in the ring $H^*(E_8/T;\mathbf{Z})$. This has been accomplished in [19, 4.2 and 6.1]. Furthermore, since $H^*(E_8/C;\mathbf{Z})$ is a direct summand of $H^*(E_8/T;\mathbf{Z})$, the elements r_{20}, r_{24} and r_{30} are not divisible by any integer in the ring $H^*(E_8/T;\mathbf{Z})$. Therefore we can replace ρ_{20}, ρ_{24} and ρ_{30} with r_{20}, r_{24} and r_{30} respectively. Summing up the results so far, we obtain the following main result of this article:

Theorem 3.4. *The integral cohomology ring of E_8/T is given as follows*:

$$H^*(E_8/T;\mathbf{Z}) = \frac{\mathbf{Z}[t_1,\dots,t_8,t,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_9,\gamma_{10},\gamma_{15}]}{\begin{pmatrix}\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_8,\rho_9,\rho_{10},\rho_{12},\rho_{14},\rho_{15},\\ \rho_{18},\rho_{20},\rho_{24},\rho_{30}\end{pmatrix}},$$

where $t_1,\dots,t_8, t \in H^2$ are as in §2, $\gamma_i \in H^{2i}$ $(i = 3,4,5,6,9,10,15)$. *The relations are*

$$\begin{aligned}
\rho_1 &= c_1 - 3t,\\
\rho_2 &= c_2 - 4t^2,\\
\rho_3 &= c_3 - 2\gamma_3,\\
\rho_4 &= c_4 + 2t^4 - 3\gamma_4,\\
\rho_5 &= c_5 - 3t\gamma_4 + 2t^2\gamma_3 - 2\gamma_5,\\
\rho_6 &= c_6 - 2\gamma_3^2 - t\gamma_5 + t^2\gamma_4 - 2t^6 - 5\gamma_6,\\
\rho_8 &= -3c_8 + 3\gamma_4^2 - 2\gamma_3\gamma_5 + t(2c_7 - 6\gamma_3\gamma_4)\\
&\quad + t^2(2\gamma_3^2 - 5\gamma_6) + 3t^3\gamma_5 + 4t^4\gamma_4 - 6t^5\gamma_3 + t^8,\\
\rho_9 &= 2c_6\gamma_3 + tc_8 + t^2c_7 - 3t^3c_6 - 2\gamma_9,\\
\rho_{10} &= \gamma_5^2 - 2c_7\gamma_3 - t^2c_8 + 3t^3c_7 - 3\gamma_{10},\\
\rho_{12} &= 15\gamma_6^2 + 2\gamma_3\gamma_4\gamma_5 - 2c_7\gamma_5 + 2\gamma_3^4 + 10\gamma_3^2\gamma_6 - 3c_8\gamma_4 - 2\gamma_4^3\\
&\quad + t(c_8\gamma_3 - 2\gamma_3^2\gamma_5 + 4c_7\gamma_4 + 6\gamma_3\gamma_4^2)\\
&\quad + t^2(3\gamma_{10} - 25\gamma_4\gamma_6 - c_7\gamma_3 - 16\gamma_3^2\gamma_4)\\
&\quad + t^3(25\gamma_3\gamma_6 - 3\gamma_4\gamma_5 + 10\gamma_3^3)\\
&\quad + t^4(3c_8 + 3\gamma_3\gamma_5 + 5\gamma_4^2)\\
&\quad + t^5(-3c_7 - 5\gamma_3\gamma_4) + 4t^6\gamma_3^2 - 7t^8\gamma_4 + 4t^9\gamma_3,\\
\rho_{14} &= c_7^2 - 3c_8\gamma_6 + 6\gamma_4\gamma_{10} - 4c_8\gamma_3^2 + 6c_7\gamma_3\gamma_4 - 6\gamma_3^2\gamma_4^2\\
&\quad - 12\gamma_4^2\gamma_6 - 2\gamma_3\gamma_5\gamma_6\\
&\quad + t(24\gamma_3\gamma_4\gamma_6 - 8c_7\gamma_3^2 - 8c_7\gamma_6 + 4c_8\gamma_5 - 6\gamma_3\gamma_{10}\\
&\quad + 12\gamma_3^3\gamma_4)\\
&\quad + t^2(-2\gamma_3\gamma_4\gamma_5 + 6\gamma_4^3 + 2\gamma_3^2\gamma_6 + 20\gamma_6^2 - 4\gamma_3^4 - c_7\gamma_5)\\
&\quad + t^3(-12\gamma_3\gamma_4^2 + 8c_8\gamma_3 - 5c_7\gamma_4 + 3\gamma_5\gamma_6)\\
&\quad + t^4(3\gamma_{10} - 26\gamma_4\gamma_6 + 6c_7\gamma_3 - 4\gamma_3^2\gamma_4)\\
&\quad + t^5(24\gamma_3\gamma_6 + 3\gamma_4\gamma_5 + 12\gamma_3^3) + t^6(-6c_8 + 2\gamma_4^2)\\
&\quad - 4t^7c_7 + t^8(6\gamma_6 - 6\gamma_3^2) - 6t^{10}\gamma_4 + 12t^{11}\gamma_3 - 2t^{14},\\
\rho_{15} &= (c_8 - t^2c_6 + 2t^3\gamma_5 + 3t^4\gamma_4 - t^8)(c_7 - 3tc_6)\\
&\quad - 2(\gamma_3^2 + c_6)(\gamma_9 - c_6\gamma_3) - 2\gamma_{15},\\
\rho_{18} &= \gamma_9^2 - 9c_8\gamma_{10} - 6\gamma_4^2\gamma_{10} - 4\gamma_3^3\gamma_9 - 10\gamma_3\gamma_6\gamma_9\\
&\quad + 2\gamma_3\gamma_5\gamma_{10} - 2\gamma_3\gamma_4\gamma_5\gamma_6 - 6c_7\gamma_3\gamma_4^2 + 3c_8\gamma_4\gamma_6\\
&\quad + c_8\gamma_3^2\gamma_4 + 6\gamma_3^2\gamma_4^3 + 12\gamma_4^3\gamma_6 + 2c_7^2\gamma_4 + 2c_7\gamma_3^2\gamma_5\\
&\quad - 2\gamma_3^3\gamma_4\gamma_5 + 2c_7\gamma_5\gamma_6 + 4\gamma_3^6 - 10\gamma_6^3 + 18\gamma_3^4\gamma_6\\
&\quad + 15\gamma_3^2\gamma_6^2 - 9c_7c_8\gamma_3\\
&\quad + t(-2\gamma_3\gamma_5\gamma_9 - 24c_7\gamma_4\gamma_6 + 8c_8\gamma_4\gamma_5 + 4c_7\gamma_3^2\gamma_4\\
&\quad + 4c_7\gamma_{10} - c_8\gamma_9 + 2c_7^2\gamma_3 + 4c_8\gamma_3\gamma_6 + 12\gamma_3\gamma_4\gamma_{10}\\
&\quad - 36\gamma_3\gamma_4^2\gamma_6 + 12\gamma_3^2\gamma_5\gamma_6 + c_8\gamma_3^3 + 6\gamma_3^4\gamma_5 - 18\gamma_3^3\gamma_4^2)\\
&\quad + t^2(24\gamma_3^4\gamma_4 - 2c_8^2 - c_7\gamma_9 - 11\gamma_3^2\gamma_{10} + 2\gamma_3\gamma_4\gamma_9\\
&\quad - 2c_8\gamma_3\gamma_5 + 16c_7\gamma_3\gamma_6 - 3c_7\gamma_4\gamma_5 + 75\gamma_4\gamma_6^2 - 6\gamma_4^4\\
&\quad - 9c_8\gamma_4^2 + 81\gamma_3^2\gamma_4\gamma_6 - 13\gamma_6\gamma_{10} + 4\gamma_3\gamma_4^2\gamma_5 - c_7\gamma_3^3)\\
&\quad + t^3(-3\gamma_5\gamma_{10} - 150\gamma_3\gamma_6^2 - 135\gamma_3^3\gamma_6 + 6\gamma_3^2\gamma_9\\
&\quad - 2c_7\gamma_3\gamma_5 + 21c_7\gamma_4^2 + 15c_7c_8 + 3\gamma_4\gamma_5\gamma_6 - 3\gamma_3^2\gamma_4\gamma_5
\end{aligned}$$

$$
\begin{aligned}
&+ 18\gamma_3\gamma_4^3 + 15\gamma_6\gamma_9 + 14c_8\gamma_3\gamma_4 - 30\gamma_3^5) \\
&+ t^4(-13c_8\gamma_6 + 2\gamma_4\gamma_{10} - 5c_7^2 - 33\gamma_3^2\gamma_4^2 + 3\gamma_5\gamma_9 \\
&- 28\gamma_3\gamma_5\gamma_6 - 45\gamma_4^2\gamma_6 - 41c_7\gamma_3\gamma_4 - 13\gamma_3^3\gamma_5 - 9c_8\gamma_3^2) \\
&+ t^5(3c_7\gamma_6 - 6\gamma_4^2\gamma_5 + 23c_7\gamma_3^2 + 105\gamma_3\gamma_4\gamma_6 - 6c_8\gamma_5 \\
&- 3\gamma_4\gamma_9 + 45\gamma_3^3\gamma_4) \\
&+ t^6(11\gamma_4^3 - 4\gamma_3\gamma_9 + 4c_7\gamma_5 + 9\gamma_3\gamma_4\gamma_5 + 12\gamma_3^4 \\
&+ 66\gamma_3^2\gamma_6 + 75\gamma_6^2 + 2c_8\gamma_4) \\
&+ t^7(-33\gamma_3\gamma_4^2 + 12\gamma_3^2\gamma_5 + 15\gamma_5\gamma_6) \\
&+ t^8(-4\gamma_{10} + 21\gamma_3^2\gamma_4 - 5c_7\gamma_3 - 3\gamma_4\gamma_6) \\
&+ t^9(6\gamma_9 - 42\gamma_3^3 - 99\gamma_3\gamma_6) \\
&+ t^{10}(-4c_8 - 6\gamma_4^2 - 13\gamma_3\gamma_5) \\
&+ t^{11}(3c_7 + 27\gamma_3\gamma_4) + t^{12}(60\gamma_6 + 18\gamma_3^2) \\
&+ 6t^{13}\gamma_5 - 9t^{14}\gamma_4 - 12t^{15}\gamma_3 + 10t^{18},
\end{aligned}
$$

$$
\begin{aligned}
\rho_{20} &= 9u^{20} + 45u^{14}v + 12u^{10}w + 60u^8v^2 + 30u^4vw \\
&\quad + 10u^2v^3 + 3w^2, \\
\rho_{24} &= 11u^{24} + 60u^{18}v + 21u^{14}w + 105u^{12}v^2 + 60u^8vw \\
&\quad + 60u^6v^3 + 9u^4w^2 + 30u^2v^2w + 5v^4, \\
\rho_{30} &= -9x^2 - 12u^9vx - 6u^5wx + 9u^{14}vw - 10u^{12}v^3 \\
&\quad - 3u^{10}w^2 + 30u^8v^2w - 35u^6v^4 + 6u^4vw^2 \\
&\quad - 10u^2v^3w - 4v^5 - 2w^3,
\end{aligned}
$$

where

$$
\begin{aligned}
u &= t_8, \\
v &= 2\gamma_6 + \gamma_3^2 - u\gamma_5 + \gamma_4(-t^2 + u^2) - u^3\gamma_3 + t^6 - t^4u^2 \\
&\quad + t^3u^3 + t^2u^4 - tu^5, \\
w &= \gamma_{10} + u\gamma_9 - u^3c_7 - u\gamma_4\gamma_5 + 2u^2\gamma_4^2 - 2u^2\gamma_3\gamma_5 \\
&\quad + \gamma_3\gamma_4(-6tu^2 + 2u^3) + \gamma_3^2(2t^2u^2 + 2tu^3 - 2u^4) \\
&\quad + \gamma_6(-5t^2u^2 + 5tu^3) + \gamma_5(t^4u + 3t^3u^2 + t^2u^3) \\
&\quad + \gamma_4(6t^4u^2 - 3t^3u^3 - 2t^2u^4 - tu^5 + u^6) \\
&\quad + \gamma_3(-6t^5u^2 - 2t^4u^3 + 4t^3u^4 + 6t^2u^5 - 4tu^6 + u^7) \\
&\quad + 4t^7u^3 - 6t^5u^5 + 2t^4u^6 + t^3u^7 - t^2u^8, \\
x &= \gamma_{15} - 20\gamma_3\gamma_6^2 + 3\gamma_3^2\gamma_9 - 23\gamma_3^3\gamma_6 - 6\gamma_3^5 + 4\gamma_6\gamma_9 \\
&\quad + 3u\gamma_4\gamma_{10} - u\gamma_5\gamma_9 - 3u\gamma_3^2\gamma_4^2 + 3uc_7\gamma_3\gamma_4 - 6u\gamma_4^2\gamma_6 \\
&\quad + (-3t + 2u)\gamma_3^3\gamma_5 + (-4t + 4u)\gamma_3\gamma_5\gamma_6 \\
&\quad + (-t^2 - u^2)\gamma_4\gamma_9 + (t^2 + tu - u^2)c_7\gamma_3^2 \\
&\quad + (9t^2 + 12tu + 5u^2)\gamma_3\gamma_4\gamma_6 \\
&\quad + (5t^2 + 6tu + 2u^2)\gamma_3^3\gamma_4 + (3t^2 + 4tu + u^2)c_7\gamma_6 \\
&\quad + (-6t^3 - 2t^2u - 6tu^2 + 5u^3)\gamma_3^4 - u^3\gamma_3\gamma_9 \\
&\quad + (3t^2u + u^3)\gamma_4^3 + (2t^2u + 3tu^2)c_7\gamma_5 \\
&\quad + (-45t^3 + 10t^2u - 40tu^2)\gamma_6^2 \\
&\quad + (t^3 - 2t^2u + tu^2 - u^3)\gamma_3\gamma_4\gamma_5 \\
&\quad + (-33t^3 + t^2u - 31tu^2 + 13u^3)\gamma_3^2\gamma_6 \\
&\quad + (-2t^4 - 4t^3u - 3tu^3 + 3u^4)c_7\gamma_4 \\
&\quad + (-9t^4 - 6t^3u - 18t^2u^2 + 5tu^3 - 3u^4)\gamma_5\gamma_6 \\
&\quad + (-3t^4 - 3t^3u - 7t^2u^2 + 5tu^3 - 4u^4)\gamma_3^2\gamma_5 \\
&\quad + (-t^4 - 6t^3u - t^2u^2 - 3tu^3)\gamma_3\gamma_4^2 \\
&\quad + (-3t^4u - 6t^3u^2 + 3t^2u^3 + 15tu^4)\gamma_{10} \\
&\quad + (-3t^4u + t^3u^2 + 5t^2u^3 + 10tu^4 - u^5)c_7\gamma_3 \\
&\quad + (15t^5 - 2t^4u + 3t^3u^2 + 14t^2u^3 - 16tu^4 + 3u^5)\gamma_3^2\gamma_4 \\
&\quad + (39t^5 - 13t^4u + 8t^3u^2 + 35t^2u^3 - 31tu^4 - 3u^5)\gamma_4\gamma_6 \\
&\quad + (t^6 - t^4u^2 - t^3u^3 - t^2u^4 - tu^5 - u^6)\gamma_9 \\
&\quad + (-13t^6 + 12t^5u + 5t^4u^2 - 56t^3u^3 + 8t^2u^4 + 21tu^5 \\
&\quad + 2u^6)\gamma_3\gamma_6 \\
&\quad + (6t^6 + 3t^5u + 2t^4u^2 + 7t^3u^3 + t^2u^4 - 8tu^5 + 3u^6)\gamma_4\gamma_5 \\
&\quad + (-8t^6 + 6t^5u + 2t^4u^2 - 22t^3u^3 + 6t^2u^4 + 8tu^5 \\
&\quad - 2u^6)\gamma_3^3 \\
&\quad + (-6t^7 + t^6u - 7t^4u^3 + 5t^3u^4 + 3t^2u^5 + 3tu^6 - 63u^7)\gamma_4^2 \\
&\quad + (-t^7 + 2t^6u + t^5u^2 - 11t^4u^3 + 6t^3u^4 + 5t^2u^5 + 6tu^6 \\
&\quad + 39u^7)\gamma_3\gamma_5 \\
&\quad + (2t^8 + 6t^7u + 3t^6u^2 - 4t^5u^3 - 15t^4u^4 + 6t^3u^5 \\
&\quad + 3t^2u^6 - 40tu^7 + 59u^8)c_7 \\
&\quad + (3t^8 + t^6u^2 + 11t^5u^3 + 14t^4u^4 - 20t^3u^5 - 4t^2u^6 \\
&\quad + 118tu^7 + 3u^8)\gamma_3\gamma_4 \\
&\quad + (-48t^9 + 3t^8u - 41t^7u^2 + 18t^6u^3 + 16t^5u^4 \\
&\quad - 13t^4u^5 - 67t^3u^6 + 125t^2u^7 - 15tu^8 - 291u^9)\gamma_6 \\
&\quad + (-18t^9 - 3t^8u - 16t^7u^2 + 10t^6u^3 - 4t^5u^4 - 8t^4u^5 \\
&\quad - 16t^3u^6 - 23t^2u^7 - 10tu^8 - 115u^9)\gamma_3^2 \\
&\quad + (-6t^{10} - 3t^9u - 9t^8u^2 + 5t^7u^3 - 5t^6u^4 - 14t^4u^6 \\
&\quad - 52t^3u^7 + 6t^2u^8 - 60tu^9 + 117u^{10})\gamma_5 \\
&\quad + (18t^{11} - 3t^{10}u + 5t^9u^2 + 11t^8u^3 - 28t^7u^4 + 8t^6u^5 \\
&\quad + 20t^5u^6 - 64t^4u^7 - 15t^3u^8 + 54t^2u^9 + 178tu^{10} \\
&\quad - 177u^{11})\gamma_4 \\
&\quad + (-2t^{12} + 6t^{11}u + 2t^{10}u^2 - 20t^9u^3 + 11t^8u^4 \\
&\quad + 22t^7u^5 - 8t^6u^6 + 83t^5u^7 + 15t^4u^8 + 5t^3u^9 \\
&\quad - 116t^2u^{10} + tu^{11} + 117u^{12})\gamma_3 \\
&\quad - 12t^{15} - t^{14}u - 10t^{13}u^2 + 6t^{12}u^3 + 7t^{11}u^4 - 13t^{10}u^5 \\
&\quad - 31t^9u^6 + 9t^8u^7 - t^7u^8 - 118t^6u^9 - 18t^5u^{10} \\
&\quad + 131t^4u^{11} - 6t^3u^{12} - 233t^2u^{13} + 175tu^{14} - 58u^{15}.
\end{aligned}
$$

4. Concluding remarks. The flag manifold G/T also appears as $G_{\mathbf{C}}/B$, where $G_{\mathbf{C}}$ denotes the complexification of G and B a Borel subgroup of $G_{\mathbf{C}}$ containing $T_{\mathbf{C}}$, the complexification of T. The subgroup B acts on $G_{\mathbf{C}}/B$ by the left translation. Each element $w \in W$ defines an element $e_w := wB$ in $G_{\mathbf{C}}/B$, and the B-orbit $X_w^\circ := B \cdot e_w$ of e_w under this action is isomorphic to an affine space, and is called a *Schu-*

bert cell corresponding to w. Then it was shown by Chevalley [9] that the manifold $G_{\mathbf{C}}/B$ admits a cellular decomposition

$$G_{\mathbf{C}}/B = \coprod_{w \in W} X_w^{\circ}. \tag{4.1}$$

The *Schubert variety* X_w corresponding to w is defined to be the closure of the Schubert cell X_w°, and it determines a *Schubert class* in $H^*(G_{\mathbf{C}}/B;\mathbf{Z})$. It follows from (4.1) that the set of Schubert classes form a $\mathbf{Z}$-basis for $H^*(G_{\mathbf{C}}/B;\mathbf{Z})$ (for details, see [1, 9, 13]). This is the *Schubert presentation* of the cohomology ring of G/T. It is natural to ask the connection between the Borel and the Schubert presentations. In [1], Bernstein, Gelfand and Gelfand introduced the *divided difference operators* and gave a general answer to the above problem. In our situation, the generators t_i $(1 \leq i \leq 8)$, t and γ_i $(i = 3,4,5,6,9,10,15)$ in Theorem 3.4 can be expressed as $\mathbf{Z}$-linear combinations of Schubert classes indexed by $W(E_8)$. The explicit forms will be given in our forthcoming paper [15] (see also [12] for the Schubert presentation of $H^*(E_8/T;\mathbf{Z})$).

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