Gröbner basis, Mordell-Weil lattices and deformation of singularities, II

By Tetsuji SHIODA*),**)

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Abstract: We prove the main theorem on the structure of everywhere integral sections on a rational elliptic surface, which is formulated in the first part of the paper with the same title [18]. A few examples are given to illustrate it, and some open questions in the case of higher arithmetic genus will be discussed.

Key words: Gröbner basis; integral section; Mordell-Weil lattice; deformation of singularities.

1. Introduction. In this paper, we prove the main theorem on the structure of everywhere integral sections on a rational elliptic surface, which is formulated in the part I [18, Theorem 2.1]. We restate the main theorem below, after recalling the necessary notation (cf. [18, §1]).

First a section of an elliptic surface $f : S \to C = \mathbf{P}^1$ is called *everywhere integral* [16] if it is disjoint from the zero-section. Denote the set of everywhere integral sections by $\mathcal{P}$ and the defining ideal of $\mathcal{P}$ by $I \subset R = k[x_j, y_k]$. By considering its primary decomposition $I = \cap_i \mathbf{q}_i$, the *multiplicity* of each $P_i \in \mathcal{P}$ is defined as $\mu(P_i) = \dim_k R/\mathbf{q}_i$ (cf. [3, 9]). Then the linear dimension $\dim_k R/I$ is equal to the number of everywhere integral sections on S counted with multiplicities.

On the other hand, given a rational elliptic surface S with the zero-section O, let $T = T_S$ denote the *trivial lattice* of S, which is embedded as a sublattice of the negative-definite root lattice E_8^-:

$$T = \oplus_{v \in R_f} T_v \subset E_8^-$$

where R_f denotes the set of $v \in C$ such that the fibre $f^{-1}(v)$ is reducible, and T_v is a sublattice of the Néron-Severi lattice $\mathrm{NS}(S)$ spanned by the irreducible components $\Theta_{v,i}$ of $f^{-1}(v)$ not intersecting the zero-section (cf. [10, 11]; also [6, 7, 19]). We sometimes view $T \subset E_8$ by changing the sign.

Now the main theorem asserts the following

Theorem 1.1. *Assume that S is a rational elliptic surface. Then* (i) *the number of everywhere integral sections is at most* 240: $0 \le n = \#\mathcal{P} \le 240$. *Moreover it is* 240 *if and only if S has no reducible fibres.*

(ii) *The linear dimension* $\dim_k R/I$ *is equal to* $240 - \nu(T)$, *where $\nu(T)$ is the number of roots in the trivial lattice T.*

(iii) *For each $i \le n$, the multiplicity $\mu(P_i)$ is equal to the combinatorial multiplicity $m(P_i)$ which is defined as the number of the distinguished roots in the root graph associated with P_i* (*see Definition* 2.2 *of* [18]).

The proof is given in the next section. In §3, a few examples are given as illustration of the main theorem. Actually rational elliptic surfaces are classified by Oguiso-Shioda [10] in terms of the trivial lattice and Mordell-Weil lattice. For every type, we have determined the data $n, m(P) (P \in \mathcal{P})$ appearing in Theorem 1.1, but the results will be reported in some other occasion. In the final section §4, we discuss some open questions in case $\chi > 1$.

2. Proof of the main theorem.

2.1. The case $T = 0$. First we consider the case $T = 0$, i.e. no reducible fibres. In this case, Theorem 1.1 reduces to the following

Theorem 2.1. *Assume that S is a rational elliptic surface with no reducible fibres. Then we have*

$$n = \dim_k R/I = 240, \ \mu(P) = m(P) = 1 \tag{2.1}$$

for all $P \in \mathcal{P}$.

Proof. We have only to prove the equality:

$$\dim R/I = 240. \tag{2.2}$$

In fact, by [18, Lemma 3.1], we know that $n = \#\mathcal{P} = 240$ and that $m(P) = 1$ for each $P \in \mathcal{P}$. The latter holds, because the root graph $\Delta(P)$ consists of the vertex $D(P)$ alone as $T = 0$. In view of the Chinese Remainder equality [18, (1.15)], we see that the claim (2.2) is equivalent to the following

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*) Research Institute for Mathematical Sciences, Kyoto University, Kitashirakawa Oiwake-cho, Sakyo-ku, Kyoto 606-8502, Japan.

**) Department of Mathematics, Rikkyo University, 3-34-1 Nishi-Ikebukuro, Toshima-ku, Tokyo 171-8501, Japan.

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(2.3) $\mu(P)=1$ for all $P\in\mathcal{P}$.

Thus we proceed as follows to show (2.2) (see Lemma 2.4 below).

First we write down a "universal" rational elliptic surface. Let S_λ denote the elliptic surface defined by the Weierstrass equation:

$$y^2+a_1(t)xy+a_3(t)y=x^3+a_2(t)x^2+a_4(t)x+a_6(t)$$

where

$$\lambda=(a_{i,j})\ (j\le i),\ a_i(t)=\sum_{j=0}^{i}a_{i,j}t^j\ (i=1,2,3,4,6).$$

Let

(2.4) $\Lambda=\{\lambda|S_\lambda \text{ is a RES}\}$

and

(2.5) $\Lambda_0=\{\lambda\in\Lambda|S_\lambda \text{ has no reducible fibres}\}$.

In characteristic different from 2 and 3, one can choose $a_i(t)=0$ $(i=1,2,3)$ (i.e. $a_{i,j}=0$ for $i=1,2,3$ and all j) without loss of generality. In any case, Λ is open in an affine space of suitable dimension, and Λ_0 is an open subset of Λ.

We denote by $\mathcal{P}_\lambda$ and I_λ the set of everywhere integral sections $\mathcal{P}$ of S_λ and its defining ideal, and by $V(I_\lambda)$ the 0-dimensional affine scheme defined by I_λ.

Lemma 2.2. *Under the assumption that* $\chi=1$, $\{V(I_\lambda)|\lambda\in\Lambda\}$ *forms a flat family over* Λ.

Proof. (I owe this remark to Takeshi Saito.) For any χ, the ideal I_λ is generated by $6\chi+1$ elements by definition, while the number of variables x_i,y_j is $(2\chi+1)+(3\chi+1)=5\chi+2$. Hence, if $\chi=1$, $V(I_\lambda)$ is a complete intersection, and the flatness follows from [4, Ch. IV]. □

Lemma 2.3. *Under the same assumption,* $\{V(I_\lambda)|\lambda\in\Lambda_0\}$ *forms a finite flat family over* Λ_0.

Proof. For any (geometric) point $\lambda\in\Lambda_0$, $V(I_\lambda)$ consists of 240 points by [18, Lemma 3.1]. The affine coordinates of these points in the ambient affine space of $V(I_\lambda)$ are given by $z(P_m)$ $(1\le m\le 240)$ (see [18, (1.7)]), if we set $\mathcal{P}_\lambda=\{P_m\ (1\le m\le 240)\}$.

Now fix any $\lambda\in\Lambda_0$. Let $\tilde\lambda$ be a generic point of Λ_0, and let $z(\tilde P)$ be a generic point of $\tilde V:=V(I_{\tilde\lambda})$. Take any specialization $\sigma:\tilde\lambda\to\lambda$, and any specialization $\tilde\sigma$ of $z(\tilde P)$ over σ. Since $\tilde V$ is specialized to $V(I_\lambda)$, bijectively as the point sets consisting of 240 points, the point $z(\tilde P)$ must specialize to one of $z(P_m)'s$, which are obviously finite. This is the case for any choice of specialization $\tilde\sigma$, and hence the family in question is a proper family (cf. [5, Ch. II] or [20, Ch. VII]). Since it is a family of 0-dimensional schemes, the assertion follows. □

Lemma 2.4. (i) *The dimension* $\dim_k R/I_\lambda$ *is constant for any* $\lambda\in\Lambda_0(k)$. (ii) *The constant value is equal to* 240.

Proof. The claim (i) follows from a general result for finite flat morphisms (see e.g. [8, Prop. 8, Lect. 6]). Thus, to prove (ii), it suffices to check it at one point $\lambda\in\Lambda_0(k)$. For instance, take λ corresponding to the rational elliptic surface $y^2=x^3+t^5+1$ treated in [18, Ex. 1.4]. (Actually, as the referee has kindly pointed out, it works only in characteristic $p\neq 2,3,5$. But we can always find a suitable equation which works in a given characteristic such as $y^2=x^3+t^5-t$ in case $p=5$.) □

This completes the proof of Theorem 2.1.

2.2. General case. Now we prove Theorem 1.1 without assumption.

For any $\lambda\in\Lambda$, let $\mathcal{D}_\lambda$ denote the set of roots in the E_8-frame on S_λ. Recall that, for each $P\in\mathcal{P}_\lambda$, $D(P):=(P)-(O)-F$ is an element of $\mathcal{D}_\lambda$.

Let $\tilde\lambda$ be a generic point of Λ_0, and let $\mathcal{P}_{\tilde\lambda}=\{\tilde P_i\ (1\le i\le 240)\}$. The set $\mathcal{D}_{\tilde\lambda}$ consists of $D(\tilde P_i)$'s by [18, Lemma 3.1].

Take any point $\lambda\in\Lambda(k)$ and any specialization $\sigma:\tilde\lambda\to\lambda$. By [18, Lemma 3.2], $\mathcal{D}_{\tilde\lambda}$ is mapped bijectively to $\mathcal{D}_\lambda$ under the specialization, and by [18, Lemma 3.3], each $D(\tilde P_i)$ is mapped either to some element of T or to an element of the form $D\equiv D(P)+\gamma$ for some $P\in\mathcal{P}_\lambda$ and $\gamma\in T$. For a fixed $P\in\mathcal{P}_\lambda$, the number of $\tilde P_i$'s corresponding to P in the above sense is equal to the multiplicity $\mu(P)$, because each $\tilde P_i$ has multiplicity 1 by (2.3) which has just been established above. Comparing this with the decomposition of the set $\mathcal{D}=\mathcal{D}_\lambda$ in [18, Theorem 3.4], we conclude that $\mu(P)=m(P)$ for each $P\in\mathcal{P}_\lambda$. This proves the claim (iii) of Theorem 1.1.

Next, to prove (ii), we note from the Chinese Remainder theorem and (iii) just proven above that

$$\dim_k R/I_\lambda=\sum_{P\in\mathcal{P}}\mu(P)=\sum_{P\in\mathcal{P}}m(P)$$

By [18, Theorem 3.4], this implies

$$\dim_k R/I_\lambda=240-\nu(T).$$

Thus we have proven the claim (ii) of Theorem 1.1.

The claim (i) is now obvious: we have

$$n=\#\mathcal{P}\le\dim_k R/I\le 240,$$

where the second inequality follows from (ii) above. This completes the proof of Theorem 1.1. □

2.3. Further information in a special case (cf. [12, 13]). The idea of the above proof is adapted from our previous work [12, §8] and [13], treating a slightly less general family which admits a singular fibre of type II (a cuspidal cubic). We remark here that, if we restrict our attention to that family, everything in the above proof becomes clearer and more explicit.

Namely we consider

$$E_\lambda : y^2 = x^3 + \left(\sum_{i=0}^{3} p_i t^i\right) x + \sum_{j=0}^{3} q_j t^j + t^5 \qquad (2.6)$$
$$\lambda = (p_0, \ldots, q_3) \in \mathbf{A}^8.$$

[From the viewpoint of topology of singularities, the above equation is called the universal deformation of E_8-singularity $y^2 = x^3 + t^5$ locally near the origin of $\mathbf{C}^3$. Our approach based on Mordell-Weil lattices is more algebraic and global, but both viewpoints are very closely connected.]

Assume that λ is generic (i.e. $p_0, \ldots, q_3$ are algebraically independent) over $\mathbf{Q}$, and let k be the algebraic closure of $k_0 := \mathbf{Q}(\lambda) = \mathbf{Q}(p_0, \ldots, q_3)$. Then the elliptic surface S_λ is a RES without reducible fibres and $M_\lambda = E_\lambda(k(t))$ is isomorphic to the root lattice E_8. Take a basis $\{P_1, \ldots . P_8\}$ forming the Dynkin diagram of type E_8, and let $u_i = sp_\infty(P_i) \in k$, where

$$sp_\infty : E_\lambda(k(t)) \to k \qquad (2.7)$$

denotes the specialization homomorphism: for any P, $sp_\infty(P)$ is defined as the unique intersection point of the section (P) and the singular fibre of type II $f^{-1}(\infty)$.

By the *fundamental theorems for the algebraic equations of type* E_8 [12, Theorems 8.3, 8.4, 8.5], we have the following results:

(i) $\mathcal{K} = \mathbf{Q}(u_1, \ldots, u_8)$ is the splitting field of $E_\lambda/\mathbf{Q}(\lambda)(t)$, i.e. we have $E_\lambda(\mathcal{K}(t)) = E_\lambda(k(t))$ and $\mathcal{K}$ is the smallest extension of $\mathbf{Q}(\lambda)$ with this property.

(ii) $\mathcal{K}/\mathbf{Q}(\lambda)$ is a Galois extension with Galois group $W(E_8)$ (the Weyl group of type E_8).

(iii) $W(E_8)$ acts on the polynomial ring $\mathbf{Q}[u_1, \ldots, u_8]$, and the ring of invariants is equal to $\mathbf{Q}[\lambda] := \mathbf{Q}[p_0, \ldots, q_3]$. In other words, $\{p_0, \ldots, q_3\}$ forms a set of fundamental invariants of $W(E_8)$ (of weight $20, 14, 8, 2, 30, 24, 18, 12$ respectively).

(iv) The minimal polynomial $\Phi(X)$ of u_1 over $\mathbf{Q}(\lambda)$ splits completely in $\mathcal{K}$ and it has coefficients in $\mathbf{Q}[\lambda]$:

$$\Phi(X, \lambda) = \prod_{i=1}^{240} (X - u_i) \in \mathbf{Q}[\lambda][X], \qquad (2.8)$$

where each root u_i is $\mathbf{Z}$-linear combination of $u_1, \ldots, u_8$. The 240 u_i form a root system of type E_8.

(v) For each $i \le 240$, there is a section $P_i \in E_\lambda(k(t))$ of the form:

$$P_i = \left(\frac{1}{u_i^2} t^2 + at + b,\ \frac{1}{u_i^3} t^3 + ct^2 + dt + e\right), \qquad (2.9)$$
$$sp_\infty(P_i) = u_i$$

where the coefficients a, b, c, d, e belong to $\mathbf{Q}(\lambda)(u_i) \cap \mathbf{Q}[u_1, \ldots, u_8]$.

Let $u := (u_1, \ldots, u_8) \in \mathbf{A}^8$. Then it follows from (iii) above that the map $\phi : u \mapsto \lambda = \phi(u)$ defines a finite ramified Galois covering $\mathbf{A}^8 \to \mathbf{A}^8$ with Galois group $W(E_8)$, which is unramified on the open set $U \subset \mathbf{A}^8$ where the "discriminant" $\delta(\lambda)$ (cf. [1]) does not vanish:

$$\delta(\lambda) = \Phi(0, \lambda) = \prod_{i=1}^{240} u_i. \qquad (2.10)$$

Furthermore $S_u := S_{\phi(u)}$ defines a *smooth* family of rational elliptic surfaces parametrized by the affine space $\mathbf{A}^8$ upstairs (see [13, Prop. 4.3] and references given there).

Now we consider specializing the generic point of the affine space upstairs $u = (u_1, \ldots, u_8)$ to some $u^0 = (u_1^0, \ldots, u_8^0)$. It induces a unique specialization $\lambda = \phi(u) \to \lambda^0 = \phi(u^0)$ in the affine space downstairs. By (v) above, we can write each P_i as $P_i(u)$ with its coefficients of t lying in $\mathbf{Q}(\lambda)(u_i) \cap \mathbf{Q}[1/u_i, u_1, \ldots, u_8]$. Hence, as far as $u_i^0 \neq 0$, P_i has a unique specialization P_i^0 with $sp_\infty(P_i^0) = u_i^0$.

Thus, if $\delta(\lambda^0) \neq 0$, $P_i \to P_i^0$ gives a bijection of the set of 240 roots in the Mordell-Weil lattice M_λ to that in M_{λ^0}. (N.B. The map $u_i \to u_i^0$ is not necessarily injective even if we assume $\delta(\lambda^0) \neq 0$. See [12, p. 685] for such an example.)

On the other hand, if $\delta(\lambda^0) = 0$, then there exist some i such that $u_i^0 = 0$. In this case, P_i must specialize to O in M_{λ^0}. The number ν of such i's is equal to $\nu(T)$, the number of roots in the trivial lattice $T \subset \mathrm{NS}(S_{\lambda^0})$. In other words, the multiplicity of the factor X in the polynomial $\Phi(X, \lambda^0)$ is equal to $\nu(T)$. If we set $\mathcal{P}_{\lambda^0} = \{Q_1, \ldots, Q_n\}$, then we have

(2.11)

$$\Phi(X,\lambda^0) = \prod_{i=1}^{240}(X - u_i^0) = X^\nu \prod_{j=1}^{n}(X - sp_\infty(Q_j))^{m(Q_j)}.$$

Thus, for a fixed $u^0 = u_i^0$, the multiplicity of $(X - u^0)$ in $\Phi(X,\lambda^0)$ is equal to the sum of $m(Q_j)$'s such that $sp_\infty(Q_j) = u^0$.

3. Examples. By [10], the Mordell-Weil lattice (abbreviated as MWL) of a rational elliptic surface is classified into 74 types by the triple $\{T, L, M\}$, where (i) T is the trivial lattice, with the opposite sign, embedded in E_8, (ii) L is the narrow MWL $E(K)^0$ which is isomorphic to the orthogonal complement of T in E_8, and (iii) M is the MWL $E(K)$ which is the direct sum of the dual lattice of L and the torsion group T'/T, where T' is the primitive closure of T in E_8.

For each type $\{T, L, M\}$, we have determined the set $\mathcal{P} \subset M$, $n = \#\mathcal{P}$, and the combinatorial multiplicities $m(P)$ for each $P \in \mathcal{P}$. The summary will be reported at some other occasion, where some **Q**-split examples (cf. [12, 14]) of every type will be given to simplify the direct verification via Gröbner basis computation (cf. [3, 9]).

Here we illustrate our results with a few classical examples. Examples in §3.1 are the prototype of the present work treated in the earlier paper [13]. Next §3.2 shows more complicated new features, dealing with the familiar Legendre curve.

3.1. Cases of higher Mordell-Weil rank (cf. [13, §5]). For a rational elliptic surface, the rank $r = M$ is bounded by 8 and the higher MW-rank cases correspond to the cases of smaller rk$T$. The first four cases in [10] (cf. [12, 13, 15]) are the following (where rk$T \leq 2$):

(i) $T = 0, L = M = E_8$,
(ii) $T = A_1, L = E_7, M = E_7^*$,
(iii) $T = A_2, L = E_6, M = E_6^*$,
(iv) $T = A_1^{\oplus 2}, L = D_6, M = D_6^*$.

The set $\mathcal{P}$ of everywhere integral sections in M consists of the roots in the root lattice L and the minimal vectors of $M = L^*$ (the dual lattice of L) for the first three cases. Thus $n = \#\mathcal{P}$ is equal to the number $\nu(L)$ of the roots in L, plus the number of minimal vectors in case (ii) or (iii): that is (i) $n = 240$, (ii) $n = 126 + 56 = 182$, (iii) $n = 72 + 54 = 126$.

If $P \in \mathcal{P}$ is a root of L, then the multiplicity $m(P)$ is 1, because the root graph consists of the single vertex $D(P)$. On the other hand, if P is a minimal vector of $M = L^*$, then the multiplicity $m(P)$ is equal to $m(P) = 2$ in case (ii) and $m(P) = 3$ in case (iii), because then the root graph $\Delta(P)$ is given, respectively, by Figure 1. Here the root $D(P)$ is denoted by the encircled vertex and other roots $\Theta_{v,i}$ in [18, (2.11)] by the black vertices. (We write θ for Θ in the following figures.)

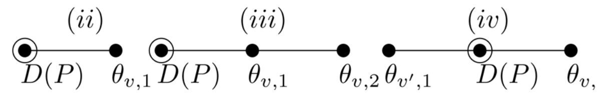

Fig. 1. Root graph $\Delta(P)$.

In case (iv), the set $\mathcal{P}$ consists of 60 roots of $L = D_6$, 12 minimal vectors of height $\langle P, P\rangle = 1$ in $M = D_6^*$, plus 64 $Q \in M$ with height $\langle Q, Q\rangle = 3/2$. We have $m(P) = 4$ and $m(Q) = 2$, as shown by Figure 1 (iv) or (ii) respectively. Compare [13, §5].

In each case, check the identity:

$$126 \cdot 1 + 56 \cdot 2 = 238 = 240 - 2,\ 2 = \nu(A_1) \tag{3.1}$$

$$72 \cdot 1 + 54 \cdot 3 = 234 = 240 - 6,\ 6 = \nu(A_2) \tag{3.2}$$

$$60 \cdot 1 + 64 \cdot 2 + 12 \cdot 4 = 236 = 240 - 4,\quad 4 = \nu(A_1^{\oplus 2}). \tag{3.3}$$

3.2. The Legendre surface. Let E be defined by the Legendre form:

$$E : y^2 = x(x-1)(x-t). \tag{3.4}$$

Let $K = k(t)$ where k is any field of characteristic $\neq 2$. The elliptic surface defined by this equation is obviously a rational surface, since the function field $K(E) = k(t, x, y)$ is equal to $k(x, y)$.

There are two singular fibres of type I_2 at $t = 0, 1$ and one of type I_2^* at $t = \infty$. The trivial sublattice $T = A_1^{\oplus 2} \oplus D_6$ is of index 4 in E_8, and the Mordell-Weil group is $M = E_8/T \cong (\mathbf{Z}/2\mathbf{Z})^2$, a torsion group of order 4. More explicitly, we have

(3.5)

$$E(K) = \{O, P_1 = (0,0), P_2 = (1,0), P_3 = (t,0)\}$$

Thus $\mathcal{P}$ consists of three 2-torsions $\{P_1, P_2, P_3\}$ and $n = \#\mathcal{P} = 3$. Figure 2 shows how each section (P_j) intersects the irreducible components $\theta_{v,i}(v = 0, 1, \infty)$ of three singular fibres. (N.B. Two different sections do not intersect. The picture is not correct in that (P_1) and (P_3) look as if they intersect.)

We can determine their (combinatorial) multiplicities as follows:

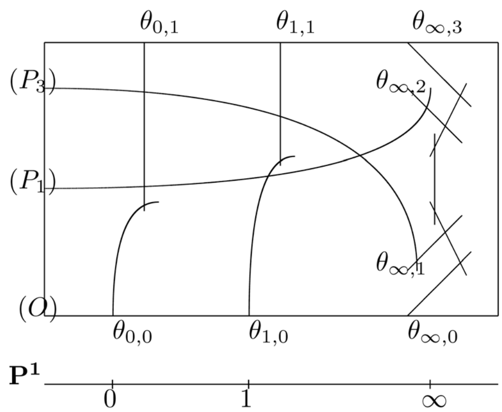

Fig. 2. Legendre elliptic surface.

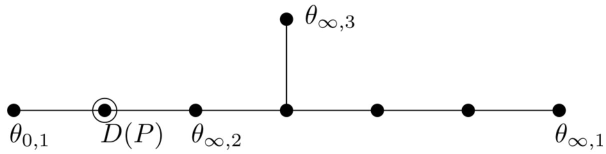

Fig. 3. Root graph $\Delta(P)$ for $P = P_1$.

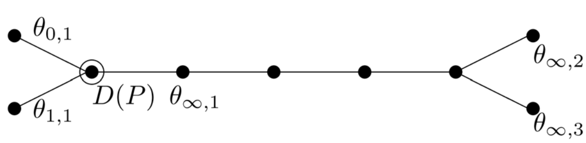

Fig. 4. Root graph $\Delta(P)$ for $P = P_3$.

(3.6) $\quad m(P_1) = 64,\ m(P_2) = 64,\ m(P_3) = 48$

Indeed the root graph $\Delta(P)$ for $P = P_1$ is shown by Figure 3 (and similarly for $P = P_2$), while $\Delta(P)$ for $P = P_3$ is as in Figure 4.

Then, by counting the number of distinguished roots in the root graph $\Delta(P)$, (3.6) can be verified. For instance, to show that $m(P_1) = 64$, consider first the distinguished roots $\xi = D(P) + \cdots$ not containing the vertex $\theta_{0,1}$ in Figure 3. Thus we seek for the number of "positive roots" in the Dynkin diagram of type E_7 whose coefficient of $D(P)$ is 1. As is well-known (see [1, 2]), there exist 33 positive roots in the Dynkin diagram of type E_7 containing the left vertex $D(P)$, but one of them is of the form $2D(P) + \cdots$. Hence we have exactly 32 ξ of the required form. Then, considering $\xi + \theta_{0,1}$ for each such ξ, we obtain another set of 32 distinguished roots. In this way, we check that the number of distinguished roots in the root graph $\Delta(P_1)$ is equal to $2 \cdot 32$, i.e. $m(P_1) = 64$.

Incidentally, it should be remarked that the root graph of an everywhere integral section P is a visual counterpart of the height formula for P. For instance, the height formula for $P = P_i$ above reads as

(3.7) $\quad \langle P_1, P_1\rangle = 2 + 0 - 6/4 - 1/2 - 0$

(3.8) $\quad \langle P_2, P_2\rangle = 2 + 0 - 6/4 - 0 - 1/2$

(3.9) $\quad \langle P_3, P_3\rangle = 2 + 0 - 1 - 1/2 - 1/2$

where the local contribution terms $contr_v(P)$ (see [11, p. 229]) on the right hand side are written in the order of $v = \infty, 0, 1$.

Now Theorem 1.1 implies that, if I denotes the defining ideal of $\mathcal{P}$ in this case, then the primary decomposition of I is of the form $I = \mathbf{q}_1 \cap \mathbf{q}_2 \cap \mathbf{q}_3$, with $\mathbf{q}_i$ corresponding to $P_i (i = 1, 2, 3)$, and we have

(3.10) $\quad \dim_k R/\mathbf{q}_i = 64 (i = 1, 2),\ \dim_k R/\mathbf{q}_3 = 48,$

implying $\dim_k R/I = 176$. As mentioned before, Gröbner basis computations allow one to make a direct verification of such a result.

4. Open questions. When the arithmetic genus χ is greater than 1, Question 1.3 posed in the Introduction of part I [18] remains open. Let us pose a few more specific questions here.

We use the same notation as in [18, §1]. In particular, $\mathcal{P}$ denotes the set of everywhere integral sections on a given elliptic surface S over $\mathbf{P}^1$ of arithmetic genus χ, and I denotes its defining ideal.

Question 4.1. *Assume that $P \in \mathcal{P}$ has height $\langle P, P\rangle = 2\chi$. Is the multiplicity $\mu(P)$ equal to 1?*

The assumption is equivalent to saying that $P \in \mathcal{P}$ belongs to the narrow Mordell-Weil lattice, or that the sections (P) and (O) intersect the same irreducible component for every reducible fibre. Question 4.1 is true if $\chi = 1$ by Theorem 1.1, since the assumption implies that the combinatorial multiplicity $m(P) = 1$.

In particular, we ask:

Question 4.2. *Assume that the trivial lattice $T = 0$, or equivalently, there are no reducible fibres. Then is it true that $I = \sqrt{I}$?*

Next consider the case $\chi = 2$, i.e. S is an elliptic K3 surface.

Question 4.3. *What is the maximum cardinality $n = \#\mathcal{P}$ when S varies among elliptic K3 surfaces?*

In characteristic 0, the largest value of n we know at the moment is $n = 5820$, attained by the elliptic K3 surface:

$$y^2 = x^3 + (t^5 - t^{-5} - 11) \tag{4.1}$$

which has MWL of rank 18 (cf. [17]).

Its reduction modulo $p = 11$ is a supersingular K3 surface, and the induced elliptic fibration has MWL of rank 20 which attains $n = 12540$. This is the largest value of n known to us.

Question 4.4. *Assume* $\chi = 2$. *Can one give some combinatorial description of the multiplicity* $\mu(P)$ *for* $P \in \mathcal{P}$?

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