Moduli of log mixed Hodge structures

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Abstract: We announce the construction of toroidal partial compactifications of the moduli spaces of mixed Hodge structures with polarized graded quotients. They are moduli spaces of log mixed Hodge structures with polarized graded quotients. We include an application to the analyticity of zero loci of normal functions.

Key words: Hodge theory; log geometry; Griffiths domain; toroidal compactification; log mixed Hodge structure; admissible normal function.

Introduction. Log Hodge structure is a natural formulation of "degenerating family of Hodge structures". In [KU09], the moduli spaces of polarized log Hodge structures were constructed. In this paper, we construct the moduli spaces of log mixed Hodge structures whose graded quotients by weight filtrations are polarized log Hodge structures. The construction is parallel to the pure case [KU09]. We add points at infinity to the non-log moduli (i.e., $\Gamma\backslash D$ for D in 1.3 and Γ in 2.1 below) by using the mixed version of nilpotent orbits. It is also parallel to the pure case to prove that the constructed spaces are actually the moduli of log mixed Hodge structures with polarized graded quotients. As in the pure case, they are like toroidal partial compactifications with slits of the moduli of mixed Hodge structures with polarized graded quotients.

We omit here the details of proofs of the above facts, which are to be published in the series of papers [KNU.p1, KNU.p2], ...

In the final section, we include an application on the analyticity of zero loci of normal functions to show how our spaces are helpful in studying such geometric problems.

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1. Mixed Hodge structures and moduli. We review the construction of the moduli spaces of mixed Hodge structures with polarized graded quotients, introduced in [U84]. This is a natural generalization of Griffiths classifying space of polarized Hodge structures [G68].

1.1. We fix a 4-ple $\Lambda = (H_0, W, (\langle\ ,\ \rangle_k)_k, ((h_k^{p,q})_{p,q})_k)$, where

H_0 is a finitely generated free $\mathbf{Z}$-module,

W is an increasing filtration on $H_{0,\mathbf{Q}} := \mathbf{Q} \otimes_{\mathbf{Z}} H_0$,

$\langle\ ,\ \rangle_k$ is a non-degenerate $\mathbf{Q}$-bilinear form $\mathrm{gr}_k^W \times \mathrm{gr}_k^W \to \mathbf{Q}$ given for each $k \in \mathbf{Z}$ which is symmetric if k is even and anti-symmetric if k is odd, and

$h_k^{p,q}$ is a non-negative integer given for $p, q, k \in \mathbf{Z}$ such that $h_k^{p,q} = 0$ unless $p+q = k$, that $h_k^{p,q} = h_k^{q,p}$ for all p, q, k, and that $\mathrm{rank}_{\mathbf{Z}}(H_0) = \sum_{p,q,k} h_k^{p,q}$, $\dim_{\mathbf{Q}}(\mathrm{gr}_k^W) = \sum_{p,q} h_k^{p,q}$ for all k.

1.2. We fix notation.

For $A = \mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$, let G_A be the group of all A-automorphisms of $H_{0,A}$ which preserve $A \otimes_{\mathbf{Z}} W$ and $A \otimes_{\mathbf{Z}} \langle\ ,\ \rangle_k$ for any k.

For $A = \mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, let $\mathfrak{g}_A$ be the set of all A-homomorphisms $N : H_{0,A} \to H_{0,A}$ satisfying the following conditions (1) and (2).

(1) $N(A \otimes_{\mathbf{Q}} W_k) \subset A \otimes_{\mathbf{Q}} W_k$ for any k.

(2) For any k, the homomorphism $\mathrm{gr}_k^W(N) : A \otimes_{\mathbf{Q}} \mathrm{gr}_k^W \to A \otimes_{\mathbf{Q}} \mathrm{gr}_k^W$ induced by N satisfies $\langle \mathrm{gr}_k^W(N)(x),\ y\rangle_k + \langle x, \mathrm{gr}_k^W(N)(y)\rangle_k = 0$ for all $x, y \in A \otimes_{\mathbf{Q}} \mathrm{gr}_k^W$.

1.3. Let D be the set of all decreasing filtration F on $H_{0,\mathbf{C}}$ for which (H_0, W, F) is a mixed Hodge structure such that the (p,q) Hodge number of $F(\mathrm{gr}_k^W)$ coincides with $h_k^{p,q}$ for any $p, q, k \in \mathbf{Z}$ and such that $F(\mathrm{gr}_k^W)$ is polarized by $\langle\ ,\ \rangle_k$ for all k.

Let $\check{D} \supset D$ be the set of all decreasing filtrations F on $H_{0,\mathbf{C}}$ satisfying the following properties (1) and (2).

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(1) $\dim \mathrm{gr}_F^p(\mathrm{gr}_{k,\mathbf{C}}^W) = h_k^{p,q}$ for any $p, q, k \in \mathbf{Z}$ such that $p + q = k$.

(2) $\langle\ ,\ \rangle_k$ kills $F^p(\mathrm{gr}_k^W) \times F^q(\mathrm{gr}_k^W)$ for any $p, q, k \in \mathbf{Z}$ such that $p + q > k$.

Then $G_{\mathbf{C}} = \mathrm{Aut}(H_{0,\mathbf{C}}, W, (\langle\ ,\ \rangle_k)_k)$ acts transitively on $\check{D}$ and hence $\check{D}$ is an analytic manifold. Furthermore, D is open in $\check{D}$ and hence it is also an analytic manifold.

2. Space of nilpotent orbits.

2.1. Set D_Σ. We fix $\Lambda = (H_0, W, (\langle\ ,\ \rangle_k)_k, (h_k^{p,q})_k)$ as in 1.1.

2.1.1. Nilpotent cone, admissibility. A finitely generated and sharp cone ([KU09] 1.3.1) of $\mathfrak{g}_{\mathbf{R}}$ is called a *nilpotent cone* if it is generated by mutually commuting nilpotent elements.

Note that, since a nilpotent cone is finitely generated, there are only finitely many faces of it.

We say a nilpotent cone σ is *admissible* [SZ85, Kas86] if it satisfies the following condition (1).

(1) For any $N \in \sigma$, the relative monodromy filtration $M(N, W)$ ([D80] (1.6.13), cf. [SZ85] (2.5)) exists. Furthermore, this filtration depends only on the smallest face of σ which contains N.

2.1.2. Fan in $\mathfrak{g}_{\mathbf{Q}}$. A *fan* Σ *in* $\mathfrak{g}_{\mathbf{Q}}$ is a set of nilpotent cones in $\mathfrak{g}_{\mathbf{R}}$ satisfying the following conditions (1)–(4).

(1) All $\sigma \in \Sigma$ are rational. That is, σ is generated by (a finite number of) elements of $\mathfrak{g}_{\mathbf{Q}}$ over $\mathbf{R}_{\geq 0}$.

(2) If $\sigma \in \Sigma$, all faces of σ belong to Σ.

(3) If $\sigma, \sigma' \in \Sigma$, $\sigma \cap \sigma'$ is a face of σ.

(4) All $\sigma \in \Sigma$ are admissible in the sense of 2.1.1.

2.1.3. Nilpotent orbit. Let D and $\check{D}$ be the spaces in 1.3.

Let σ be an admissible nilpotent cone. A subset Z of $\check{D}$ is called a *σ-nilpotent orbit* (resp. *σ-nilpotent i-orbit*) if the following conditions (1)–(3) are satisfied for some $F \in Z$.

(1) $Z = \exp(\sigma_{\mathbf{C}})F$ (resp. $Z = \exp(i\sigma_{\mathbf{R}})F$). Here $\sigma_{\mathbf{C}}$ (resp. $\sigma_{\mathbf{R}}$) is the $\mathbf{C}$ (resp. $\mathbf{R}$)-linear span of σ in $\mathfrak{g}_{\mathbf{C}}$ (resp. $\mathfrak{g}_{\mathbf{R}}$).

(2) $N(F^p) \subset F^{p-1}$ for all $N \in \sigma$ and $p \in \mathbf{Z}$.

(3) Take a finite family $(N_j)_{1 \leq j \leq n}$ of elements of σ which generates σ. Then, if $y_j \in \mathbf{R}$ and y_j are sufficiently large for $1 \leq j \leq n$, we have $\exp(\sum_{j=1}^n iy_jN_j)F \in D$.

Note that, if (1)–(3) are satisfied for some $F \in Z$, then they are satisfied for all $F \in Z$. See [KNU08] 12.10 for a review on how the admissibility appears in geometry.

The above notion of nilpotent orbit is closely related to the notion of infinitesimal mixed Hodge module (IMHM) in [Kas86] (see [KNU08] 5.2). For a nilpotent cone σ generated by $N_1, \ldots, N_n$ and for $F \in \check{D}$, $(H_{0,\mathbf{C}}; W_{\mathbf{C}}; F, \bar{F}; N_1, \ldots, N_n)$ is an IMHM if and only if σ is admissible and $\exp(\sigma_{\mathbf{C}})F$ is a σ-nilpotent orbit. Here $\bar{F}$ denotes the complex conjugate of F.

2.1.4. Let D_Σ (resp. $D_\Sigma^\sharp$) be the set of all pairs (σ, Z), where $\sigma \in \Sigma$ and Z is a σ-nilpotent orbit (resp. σ-nilpotent i-orbit).

We have embeddings $D \subset D_\Sigma$ and $D \subset D_\Sigma^\sharp$, $F \mapsto (\{0\}, \{F\})$.

We have a surjection $D_\Sigma^\sharp \to D_\Sigma, (\sigma, Z) \mapsto (\sigma, \exp(\sigma_{\mathbf{C}})Z)$.

2.1.5. Compatibility with Γ. Let Γ be a subgroup of $G_{\mathbf{Z}}$. We say Σ and Γ are *compatible*, if for any $\gamma \in \Gamma$ and $\sigma \in \Sigma$, we have $\mathrm{Ad}(\gamma)\sigma \in \Sigma$.

Then Γ acts on D_Σ and also on $D_\Sigma^\sharp$ by $(\sigma, Z) \mapsto (\mathrm{Ad}(\gamma)\sigma, \gamma Z)$ $(\gamma \in \Gamma)$.

Further, we say Σ and Γ are *strongly compatible* if they are compatible and, for any $\sigma \in \Sigma$, any element of σ can be written as a finite sum of elements of the form $a \log(\gamma)$, where $a \in \mathbf{R}_{\geq 0}$ and $\gamma \in \Gamma(\sigma) := \Gamma \cap \exp(\sigma)$.

2.2. Set E_σ. Let Σ and Γ be as in 2.1. Assume that they are strongly compatible. We fix $\sigma \in \Sigma$ in this 2.2.

2.2.1. Let $D_\sigma = D_{\mathrm{face}(\sigma)}$, $D_\sigma^\sharp = D_{\mathrm{face}(\sigma)}^\sharp$, where face($\sigma$) denotes the fan consisting of all faces of σ.

Since $\Gamma(\sigma)$ (2.1.5) is a sharp and torsion free fs monoid ([Kat89], [KU09] 2.1.4), the associated group $\Gamma(\sigma)^{\mathrm{gp}}$ is a finitely generated free abelian group and it is strongly compatible with the fan face(σ).

We will regard $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ as a quotient of a subset E_σ of an analytic space $\check{E}_\sigma$ explained below (see [KU09] Ch.3, Ch.4 for the pure version).

2.2.2. Associated to σ, we have the torus and the toric variety: $\mathrm{torus}_\sigma := \mathrm{Spec}(\mathbf{C}[\Gamma(\sigma)^{\vee\,\mathrm{gp}}])_{\mathrm{an}} \subset \mathrm{toric}_\sigma := \mathrm{Spec}(\mathbf{C}[\Gamma(\sigma)^\vee])_{\mathrm{an}}$. Here and hereafter we denote $\mathrm{Hom}(P, \mathbf{N})$ by $P^\vee$ for an fs monoid P.

We denote

$$\check{E}_\sigma = \mathrm{toric}_\sigma \times \check{D}.$$

We use some facts about log structures in [Kat89, KU09]. We endow toric_σ with the canonical fs log structure (cf. [KU09] 2.1.6 (ii)), and endow $\check{E}_\sigma$ with its inverse image. Since $\check{D}$ is smooth, $\check{E}_\sigma$ is a logarithmically smooth fs log analytic space ([Kat89], cf. also [KU09] 2.1.11).

Let $(\mathrm{toric}_\sigma^{\log}, \mathcal{O}^{\log}_{\mathrm{toric}_\sigma})$ be the associated ringed space ([KN99], cf. also [KU09] 2.2). We have isomorphisms $\pi_1(\mathrm{toric}_\sigma^{\log}) \simeq \pi_1(\mathrm{torus}_\sigma) \simeq \Gamma(\sigma)^{\mathrm{gp}}$ (cf. [KU09] 3.3.2).

2.2.3. Let $q \in \mathrm{toric}_\sigma$. Let $\sigma(q)$ be the face of σ corresponding to the face $\pi_1^+(q^{\log}) := \pi_1(q^{\log}) \cap \Gamma(\sigma)$ of $\Gamma(\sigma)$. Let $\mathcal{S}' = \{f \in \Gamma(\sigma)^\vee \mid f(q) \neq 0\}$, where f is regarded as a holomorphic function on toric_σ. Let $\mathbf{e} : \sigma_{\mathbf{C}}/(\sigma(q)_{\mathbf{C}} + \log(\Gamma(\sigma)^{\mathrm{gp}})) \xrightarrow{\sim} \mathrm{Hom}((\mathcal{S}')^{\mathrm{gp}}, \mathbf{C}^\times)$ be the isomorphism defined by $(\mathbf{e}(z \log \gamma), f) = \exp(2\pi i z(\gamma, f))$ for $z \in \mathbf{C}$, $\gamma \in \Gamma(\sigma)^{\mathrm{gp}}$, $f \in (\mathcal{S}')^{\mathrm{gp}}$. Let z be an element of $\sigma_{\mathbf{C}}$ whose image in $\sigma_{\mathbf{C}}/(\sigma(q)_{\mathbf{C}} + \log(\Gamma(\sigma)^{\mathrm{gp}}))$ coincides with the class of $q \in \mathrm{Hom}((\mathcal{S}')^{\mathrm{gp}}, \mathbf{C}^\times)$ under the above isomorphism.

We define the subset E_σ of $\check{E}_\sigma$ by the following condition.

For $(q, F) \in \check{E}_\sigma = \mathrm{toric}_\sigma \times \check{D}$, $(q, F) \in E_\sigma$ if and only if $\exp(\sigma(q)_{\mathbf{C}}) \exp(z)F$ is a $\sigma(q)$-nilpotent orbit.

Denote $|\,\mathrm{toric}\,|_\sigma := \mathrm{Hom}(\Gamma(\sigma)^\vee, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \subset \mathrm{toric}_\sigma = \mathrm{Hom}(\Gamma(\sigma)^\vee, \mathbf{C}^{\mathrm{mult}})$, where $\mathbf{R}^{\mathrm{mult}}_{\geq 0}$ and $\mathbf{C}^{\mathrm{mult}}$ are the sets $\mathbf{R}_{\geq 0}$ and $\mathbf{C}$ regarded as monoids by multiplication, respectively. Define

$$\check{E}^\sharp_\sigma = |\,\mathrm{toric}\,|_\sigma \times \check{D} \subset \check{E}_\sigma, \quad E^\sharp_\sigma = E_\sigma \cap \check{E}^\sharp_\sigma.$$

Then the subset $E^\sharp_\sigma$ of $\check{E}^\sharp_\sigma$ can be characterized by the following condition.

For $(q, F) \in \check{E}^\sharp_\sigma = |\,\mathrm{toric}\,|_\sigma \times \check{D}$, $(q, F) \in E^\sharp_\sigma$ if and only if $\exp(i\sigma(q)_{\mathbf{R}}) \exp(iy)F$ is a $\sigma(q)$-nilpotent i-orbit. Here $y \in \mathbf{R}$ is the imaginary part of the above z.

2.2.4. Define canonical maps $\varphi : E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ and $\varphi^\sharp : E^\sharp_\sigma \to D^\sharp_\sigma$ by

$$\varphi(q, F) = ((\sigma(q), \exp(\sigma(q)_{\mathbf{C}}) \exp(z)F) \mod \Gamma(\sigma)^{\mathrm{gp}}),$$

$$\varphi^\sharp(q, F) = (\sigma(q), \exp(i\sigma(q)_{\mathbf{R}}) \exp(iy)F).$$

2.3. Strong topology, the category $\mathcal{B}$(log), log manifolds. In 2.4 below, we will endow $\Gamma\backslash D_\Sigma$ with a structure of a local ringed space over $\mathbf{C}$ with an fs log structure, and we define a topology on $D^\sharp_\Sigma$. In this 2.3, we give preparation for them.

2.3.1. Strong topology. Let Z be an analytic space, and S a subset. The *strong topology* of S in Z is defined as follows. A subset U of S is open for this topology if and only if for any analytic space A and any morphism $f : A \to Z$ of analytic spaces such that $f(A) \subset S$, $f^{-1}(U)$ is open in A. It is stronger than or equal to the topology as a subspace of Z.

2.3.2. The category $\mathcal{B}$(log). As in [KU09], in the theory of moduli spaces of log mixed Hodge structures, we have to enlarge the category of analytic spaces because the moduli spaces are often not analytic spaces.

Let $\mathcal{A}$ be the category of anaytic spaces and let $\mathcal{A}$(log) be the category of fs log analytic spaces. We enlarge $\mathcal{A}$ and $\mathcal{A}$(log) to $\mathcal{B}$ and $\mathcal{B}$(log), respectively, as follows:

$\mathcal{B}$ (resp. $\mathcal{B}$(log)) is the category of all local ringed spaces S over $\mathbf{C}$ (resp. local ringed spaces S over $\mathbf{C}$ endowed with an fs log structure) having the following property: S is locally isomorphic to a subset of an analytic space (resp. an fs log analytic space) Z with the strong topology in Z (2.3.1) and with the inverse image (resp. inverse images) of the sheaf of rings $\mathcal{O}_Z$ (resp. $\mathcal{O}_Z$ and the log structure M_Z).

2.3.3. A log manifold. A *log manifold* is an object S of $\mathcal{B}$(log) such that locally on S, we can take a logarithmically smooth fs log analytic space Z and $\omega_1, \cdots, \omega_n \in \Gamma(Z, \omega^1_Z)$ (ω^1_Z is the sheaf of differential forms with log poles; see [KU09] 2.1.7) such that S (as an object of $\mathcal{B}$(log)) is isomorphic to an open subspace of the subspace $\{z \in Z \mid \omega_1, \cdots, \omega_n$ are zero in $\omega^1_z\}$ with the strong topology in Z and the inverse images of $\mathcal{O}_Z$ and M_Z.

In [KU09] 4.1.1, we saw that moduli spaces of polarized log Hodge structures were log manifolds (loc. cit. 3.5.7, 3.5.8).

2.4. Topology, local ringed space structure, log structure of $\Gamma\backslash D_\Sigma$. Let Σ be a fan in $\mathfrak{g}_{\mathbf{Q}}$ (2.1.2). Let Γ be a subgroup of $G_{\mathbf{Z}}$, which is strongly compatible with Σ. Let $\sigma \in \Sigma$.

2.4.1. We endow the subset E_σ of $\check{E}_\sigma$ in 2.2 with the following structures of log local ringed spaces over $\mathbf{C}$. The topology is the strong topology in $\check{E}_\sigma$. The sheaf $\mathcal{O}$ of rings and the log structure M are the inverse images of $\mathcal{O}$ and M of $\check{E}_\sigma$, respectively.

We endow $\Gamma\backslash D_\Sigma$ with the strongest topology for which the maps $\pi_\sigma : E_\sigma \xrightarrow{\varphi} \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \to \Gamma\backslash D_\Sigma$ are continuous for all $\sigma \in \Sigma$. Here φ is as in 2.2.4. We endow $\Gamma\backslash D_\Sigma$ with the following sheaf of rings $\mathcal{O}_{\Gamma\backslash D_\Sigma}$ over $\mathbf{C}$ and the following log structure $M_{\Gamma\backslash D_\Sigma}$. For any open set U of $\Gamma\backslash D_\Sigma$ and for any $\sigma \in \Sigma$, let $U_\sigma := \pi_\sigma^{-1}(U)$ and define $\mathcal{O}_{\Gamma\backslash D_\Sigma}(U)$ (resp. $M_{\Gamma\backslash D_\Sigma}(U)) := \{\text{map } f : U \to \mathbf{C} \mid f \circ \pi_\sigma \in \mathcal{O}_{E_\sigma}(U_\sigma)(\text{resp.} \in M_{E_\sigma}(U_\sigma))(\forall \sigma \in \Sigma)\}$.

2.4.2. We introduce the topology of $E^\sharp_\sigma$ as a subspace of E_σ (2.2.3). We introduce on $D^\sharp_\Sigma$ the strongest topology for which the maps $E^\sharp_\sigma \xrightarrow{\varphi^\sharp} D^\sharp_\sigma \to D^\sharp_\Sigma$ ($\sigma \in \Sigma$) are continuous. Here $\varphi^\sharp$ is as in 2.2.4. Note that the surjection $D^\sharp_\Sigma \to \Gamma\backslash D_\Sigma$ (cf. 2.1.4) becomes continuous.

2.4.3. The above topologies have the following properties.

(1) Let $(\sigma, Z) \in D_\Sigma$, let $F \in Z$, and write $\sigma = \sum_{1\le j\le n} \mathbf{R}_{\ge 0} N_j$. Then

$$((\sigma, Z) \bmod \Gamma) = \lim_{\substack{\mathrm{Im}(z_j)\to\infty \\ 1\le j\le n}} (\exp(\textstyle\sum_{1\le j\le n} z_j N_j)F \bmod \Gamma) \quad \text{in } \Gamma\backslash D_\Sigma.$$

(2) Let $(\sigma, Z) \in D^\sharp_\Sigma$, let $F \in Z$, and let N_j be as above. Then

$$(\sigma, Z) = \lim_{\substack{y_j\to\infty \\ 1\le j\le n}} \exp(\textstyle\sum_{1\le j\le n} iy_j N_j)F \quad \text{in } D^\sharp_\Sigma.$$

Theorem A. *Let Σ be a fan in $\mathfrak{g}_{\mathbf{Q}}$* (2.1.2). *Let Γ be a subgroup of $G_{\mathbf{Z}}$, which is strongly compatible with Σ. Let $\sigma \in \Sigma$.*

(1) *E_σ belongs to $\mathcal{B}$*(log). *It is a log manifold.*

(2) *$E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ is a $\sigma_{\mathbf{C}}$-torsor in the category $\mathcal{B}$*(log). *$E^\sharp_\sigma \to D^\sharp_\sigma$ is an $i\sigma_{\mathbf{R}}$-torsor in the category of topological spaces.*

(3) *The action of Γ on $D^\sharp_\Sigma$ is proper. The spaces $\Gamma\backslash D^\sharp_\Sigma$ and $\Gamma\backslash D_\Sigma$ are Hausdorff.*

(4) *Assume that Γ is neat. Then $D^\sharp_\Sigma \to \Gamma\backslash D^\sharp_\Sigma$ is a local homeomorphism.*

(5) *Assume that Γ is neat. Then $\Gamma\backslash D_\Sigma$ belongs to $\mathcal{B}$*(log). *It is a log manifold and $(\Gamma\backslash D_\Sigma)^{\log} = \Gamma\backslash D^\sharp_\Sigma$.*

For the notion "neat", see [B69], cf. also [KU09] 0.4.1.

3. Log mixed Hodge structures and moduli. Let S be an object in $\mathcal{B}$(log). Then we have the associated ringed space $(S^{\log}, \mathcal{O}^{\log}_S)$. See [KU09] 2.2 for the definition.

3.1. Log mixed Hodge structures.

3.1.1. Polarized log Hodge structure. A polarized log Hodge structure was the main ingredient of [KU09]. See 2.4 in loc. cit. for the definition.

The definition includes two points.

(1) PHS after sufficiently twisted specializations.

(2) Small (i.e., pointwise) Griffiths transversality.

3.1.2. For example, degeneration of elliptic curves gives a polarizable pure log Hodge structure of weight 1 (not mixed!). This is explained in detail in [KU09] §0.

3.1.3. Log mixed Hodge structure with polarized graded quotients. This is the main ingredient of this paper. Let S be an object in $\mathcal{B}$(log). A *log mixed Hodge structure with polarized graded quotients* (*LMH with PGQ*, for short) on S is a 4-ple $(H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_k)_k, F)$, where

$H_{\mathbf{Z}}$ is a locally constant sheaf of finitely generated free $\mathbf{Z}$-modules on $S^{\log}$,

W is an increasing filtration on $H_{\mathbf{Q}} := \mathbf{Q} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$,

$\langle\ ,\ \rangle_k$ is a $(-1)^k$-symmetric bilinear form $\mathrm{gr}^W_k \times \mathrm{gr}^W_k \to \mathbf{Q}$ given for each $k \in \mathbf{Z}$,

F is a decreasing filtration of the $\mathcal{O}^{\log}_S$-module $\mathcal{O}^{\log}_S \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$,

such that $(H_{\mathbf{Z}}, W_{\mathbf{R}}, F)$ is an LMH in the sense of [KU09] 2.6 and such that for each $k \in \mathbf{Z}$, the induced data on gr^W_k form a polarized log Hodge structure of pure weight k.

Thus there are three points in the definition.

(1) MHS with polarized graded quotients after sufficiently twisted specializations.

(2) Small Griffiths transversality.

(3) Admissibility of local monodromy.

3.1.4. The key observation in the pure case in [KU09] 0.4.25 can be generalized, and we have also in the present case the following

(an LMH with PGQ on an fs log point)

= (a nilpotent orbit in the mixed case).

3.2. Moduli functor. We define the moduli functor of log mixed Hodge structure with polarized graded quotients.

3.2.1. Fix $\Phi = (\Lambda, \Sigma, \Gamma)$, where Λ is as in 1.1, Σ and Γ are as in 2.1, and Σ is assumed to be strongly compatible with Γ.

3.2.2. Let S be an object of $\mathcal{B}$(log). By *a log mixed Hodge structure of type* Φ on S, we mean an LMH with polarized graded quotients $H = (H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_k)_k, F)$ endowed with a global section μ of the sheaf $\Gamma\backslash\mathrm{Isom}((H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_k)_k), (H_0, W, (\langle\ ,\ \rangle_k)_k))$ on $S^{\log}$ which satisfies the following conditions (1) and (2).

(1) $\mathrm{rank}_{\mathbf{Z}}(H_{\mathbf{Z}}) = \sum_{p,q,k} h^{p,q}_k$, $\quad \mathrm{rank}_{\mathcal{O}^{\log}_S}(F^p) = \sum_{k\in\mathbf{Z}, r\ge p} h^{r,k-r}_k$ for all p.

(2) For any $s \in S$ and $t \in S^{\log}$ lying over s, if $\tilde{\mu}_t : (H_{\mathbf{Z},t}, W, (\langle\ ,\ \rangle_k)_k) \xrightarrow{\sim} (H_0, W, (\langle\ ,\ \rangle_k)_k)$ is a representative of the germ of μ at t, then there exists $\sigma \in \Sigma$ such that the image of the composite map $\mathrm{Hom}(M_{S,s}/\mathcal{O}^\times_{S,s}, \mathbf{N}) \hookrightarrow \pi_1(\tau^{-1}(s)) \to \mathrm{Aut}(H_{\mathbf{Z},t}, W, (\langle\ ,\ \rangle_k)_k) \xrightarrow{\text{by } \tilde{\mu}_t} \mathrm{Aut}(H_0, W, (\langle\ ,\ \rangle_k)_k)$ is contained in $\exp(\sigma)$. Furthermore, if we take the smallest such $\sigma \in \Sigma$, then the $\exp(\sigma_{\mathbf{C}})$-orbit Z including $\tilde{\mu}_t(\mathbf{C} \otimes_{\mathcal{O}^{\log}_{S,t}} F_t)$, which is independent of the choice of a $\mathbf{C}$-algebra homomorphism $\mathcal{O}^{\log}_{S,t} \to \mathbf{C}$, is a σ-nilpotent orbit (cf. 3.1.4, cf. also [KU09] 0.4.24, 2.5.1, 2.5.5).

We call a log mixed Hodge structure of type Φ on S also a *log mixed Hodge structure with polarized graded quotients, with global monodromy in* Γ, *and with local monodromy in* Σ.

3.2.3. Let $\mathrm{LMH}_\Phi : \mathcal{B}(\log) \to (\mathrm{set})$ be the contravariant functor defined as follows: $\mathrm{LMH}_\Phi(S)$ for an object S of $\mathcal{B}(\log)$ is the set of isomorphism classes of log mixed Hodge structures of type Φ on S.

Theorem B. *Assume that* Γ *is neat. Then the functor* LMH_Φ *in* 3.2.3 *is represented by* $\Gamma\backslash D_\Sigma$.

The period map $\mathrm{LMH}_\Phi \to \mathrm{Mor}(\bullet, \Gamma\backslash D_\Sigma)$ which is the isomorphism in this Theorem B is as follows. Let S and H be as in 3.2.2. Let $s \in S$. The associated point of $\Gamma\backslash D_\Sigma$ by this period map is the image of $(\sigma, Z) \in D_\Sigma$ in $\Gamma\backslash D_\Sigma$, where σ and Z are the ones in the last sentence of 3.2.2 (2).

The proofs of the theorems so far in this paper are similar to those in the pure case [KU09]. In the pure case, the key tool is the SL(2)-orbit theorem in several variables of Cattani-Kaplan-Schmid [CKS86]. Instead of this, here we use a mixed Hodge theoretic version [KNU08] of the SL(2)-orbit theorem in several variables.

4. Some application. Let the notation be as in §1.

Theorem. *Let* S *be a complex analytic manifold, let* T *be a smooth divisor on* S*, and let* $S^* = S - T$. *Let* Γ *be a neat subgroup of* $G_{\mathbf{Z}}$. *For* $1 \le j \le n$, *let* $f_j : S^* \to \Gamma\backslash D$ *be the period maps associated to some variations of mixed Hodge structure* H_j *with polarized graded quotients which are admissible with respect to* S.

Let $V = \{s \in S^* \mid f_1(s) = \cdots = f_n(s)\}$, *and let* $\bar{V}$ *be the closure of* V *in* S. *Then* $\bar{V}$ *is an analytic subset of* S.

Proof. By a standard argument, we may assume that the local monodromy of H_j along the divisor T is unipotent. Then, by the admissibility, H_j extends to a log mixed Hodge structure $\tilde{H}_j$ on S (see [KNU08] §12). Hence the period map f_j of H_j extends to a morphism $\bar{f}_j : S \to \Gamma\backslash D_\Xi$ in $\mathcal{B}(\log)$ corresponding to $\tilde{H}_j$, where Ξ is the fan consisting of all the one-dimensional rational nilpotent cones and $\{0\}$ (see [KU09] 4.3.1 (i) for the pure case). To prove the Theorem, it is sufficient to see that $\{s \in S \mid \bar{f}_1(s) = \cdots = \bar{f}_n(s)\}$ is a closed analytic subset of S. Hence we are reduced to:

Proposition. *Let* S *be a complex analytic space, let* Y *be an object of* $\mathcal{B}$, *let* $f_j : S \to Y$ $(1 \le j \le n)$ *be morphisms in* $\mathcal{B}$, *and assume that* Y *is Hausdorff as a topological space. Let* $C = \{s \in S \mid f_1(s) = \cdots = f_n(s)\}$. *Then* C *is a closed analytic subset of* S.

Proof. Let $f = (f_j)_j : S \to Y^n$. Working locally on Y, we may assume that Y is a subset of a complex analytic space Z and $\mathcal{O}_Y$ is the inverse image of $\mathcal{O}_Z$. Then C is the fiber product of $S \to Z^n \leftarrow Z$, where $S \to Z^n$ is the composite $S \to Y^n \to Z^n$ and $Z \to Z^n$ is the diagonal morphism. This proves Proposition. $\square$

Remark 1. The above theorem yields an alternative proof of the analyticity of the closure of the zero locus of an admissible normal function for a family of intermediate Jacobians over S^*, which was proved by Saito in [Sa.p] and by Brosnan-Pearlstein in [BP.p1]. This is possible in virtue of the facts that the universal intermediate Jacobian appears as the graded quotient map $\Gamma\backslash D \to \Gamma'\backslash D'$ for the type explained below, where D' and Γ' are D and Γ for gr^W, respectively, and that an admissible normal function is regarded as a period map $S^* \to \Gamma\backslash D$.

More precisely, let H' be a variation of polarized Hodge structure of weight -1 on S^*. Assume that H' has unipotent monodromy. Let $J(H')$ be the intermediate Jacobian. Fix $s \in S^*$, and let $H_0 := H'_{\mathbf{Z},s} \oplus \mathbf{Z}$, $W_{-1} = H'_{\mathbf{Q},s}$, $W_0 = H_{0,\mathbf{Q}}$, $\langle\, ,\, \rangle_0 : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Q}; (a, b) \mapsto ab$, $\langle\, ,\, \rangle_{-1}$ and the $h^{p,q}_{-1}$ are the ones determined by H', $h^{0,0}_0 = 1$, and $h^{p,q}_k = 0$ for the other p, q. Let $\Gamma' \subset \mathrm{Aut}_{\mathbf{Z}}(H'_{\mathbf{Z},s})$ be the monodromy group of H', and let Γ be the subgroup of $G_{\mathbf{Z}}$ consisting of all the elements whose restrictions to $\mathrm{gr}^W_{-1}(H_0)$ belong to Γ' and which induce 1 on $\mathrm{gr}^W_0(H_0) = \mathbf{Z}$. Then $J(H') \subset S^* \times \Gamma\backslash D$ (cf. [KNU10]), and an admissible normal function $\nu : S^* \to J(H')$ is identified with the composite $f : S^* \to \Gamma\backslash D$ of the second projection after ν. The last map is nothing but the period map associated to H in the extension $0 \to H' \to H \to \mathbf{Z} \to 0$ of variations of mixed Hodge structure corresponding to ν. If admissible normal functions $\nu_j : S^* \to J(H')$ correspond to f_j in Theorem, we have $V = \{s \in S^* \mid \nu_1(s) = \cdots = \nu_n(s)\}$. Assume further that $n = 2$ and ν_2 is the zero section of $J(H')$ over S^*. Then, V is the zero locus of ν_1 in S^* and Theorem asserts that its closure in S is analytic, which is proved in [Sa.p] and in [BP.p1].

Remark 2. In the series of papers [KNU.p1, KNU.p2], ..., we plan to give a generalization of the Theorem in which T can be any closed analytic subspace of S. This will give an alternative proof of a result of Brosnan and Pearlstein ([BP.p2]; cf. also [Sc.p] by Schnell) when V is the zero locus of an admissible normal function.

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