Analyticity and smoothing effect for the fifth order KdV type equation

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Abstract: We consider the initial value problem for the reduced fifth order KdV type equation: $\partial_t u - \partial_x^5 u - 10\partial_x(u^3) + 5\partial_x(\partial_x u)^2 = 0$ which is obtained by removing the nonlinear term $10\partial_x(u\partial_x^2 u)$ from the fifth order KdV equation. We show the existence of the local solution which is real analytic in both time and space variables, if the initial data $\phi \in H^s(\mathbf{R})$ $(s > 1/8)$ satisfies the condition

$$\sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|(x\partial_x)^k \phi\|_{H^s} < \infty,$$

for some constant $A_0 (0 < A_0 < 1)$. Moreover, the smoothing effect for this equation is obtained. The proof of our main result is based on the argument used in [5].

Key words: Analytic; smoothing effect; fifth order KdV equation; KdV hierarchy.

1. Introduction. The KdV hierarchy is well known as the series of the Lax pair formulation [7,8], which are presented as

$$(1.1)_0 \qquad \partial_t u - \partial_x u = 0,$$

$$(1.1)_1 \qquad \partial_t u + \partial_x^3 u - 6u\partial_x u = 0,$$

$$(1.1)_2 \qquad \partial_t u - \partial_x^5 u - 10\partial_x(u^3) + 5\partial_x(\partial_x u)^2 + 10\partial_x(u\partial_x^2 u) = 0.$$

$$\vdots$$

We are interested in the existence theory of the analytic solution and the smoothing effect of the KdV hierarchy. T. Kato-Masuda [6] proved the existence of the analytic solution in the space variable for the generalized KdV equation. K. Kato-Ogawa [5] proved that $(1.1)_1$ has the real analytic solution in both time and space variables and the smoothing effect. Recently, it is shown that the nonlinear dispersive equations including the KdV hierarchy has the local analytic solution in the space variable (see [4]). However, neither the existence of the real analytic solution in both time and space variables nor the smoothing effect is obtained for $(1.1)_2$.

On the other hand, we may expect that the method used in [5] can work for the reduced equations given by removing some nonlinear terms from the higher order KdV equations $(1.1)_j$ with $j \geq 2$.

In this paper, as a starting point for this attempt, we consider the following initial value problem of the reduced fifth order KdV type equation:

$$(1.2) \quad \begin{cases} \partial_t u - \partial_x^5 u = \partial_x(u^3) + \partial_x(\partial_x u)^2, & t, x \in \mathbf{R}, \\ u(0,x) = \phi(x), & x \in \mathbf{R}, \end{cases}$$

where we may take all coefficients of the nonlinear terms to be equal to 1 without loss of generality. This equation is obtained by removing the nonlinear term $10\partial_x(u\partial_x^2 u)$ from the original fifth order KdV equation $(1.1)_2$. Our main purpose is to prove not only the existence of a local real analytic solution of (1.2) in both time and space variables but also the smoothing effect.

Before stating the main result precisely, we introduce the function space introduced by Bourgain (see [2]). For $s, b \in \mathbf{R}$,

$$X_b^s = \{f \in \mathcal{S}'(\mathbf{R}^2);\ \|f\|_{X_b^s} < \infty\},$$

where

$$\|f\|_{X_b^s}^2 = \iint_{\mathbf{R}^2} (1 + |\tau - \xi^5|)^{2b} (1 + |\xi|)^{2s} |\hat{\hat{f}}(\tau,\xi)|^2 d\tau d\xi,$$

and $\hat{\hat{f}}$ (or $\mathcal{F}_{t,x} f$) is the Fourier transform of f in both x and t variables; that is,

$$\hat{\hat{f}}(\tau,\xi) = (2\pi)^{-1} \iint_{\mathbf{R}^2} f(t,x) e^{-it\tau - ix\xi} dt dx.$$

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Our main result is described in the following

Theorem 1.1. *Let $s > 1/8$ and let $b \in (1/2, 23/40)$. Then for any $\phi(x) \in H^s(\mathbf{R})$ such that*

$$(1.3)\quad \begin{cases} (x\partial_x)^k \phi(x) \in H^s(\mathbf{R}) \quad (k = 0, 1, 2, \cdots), \\ \displaystyle\sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|(x\partial_x)^k \phi\|_{H^s} < \infty \\ \qquad \text{for some constant } A_0 \ (0 < A_0 < 1), \end{cases}$$

there exist a constant $T = T(\phi) > 0$ and a unique solution $u \in C((-T, T), H^s) \cap X_b^s$ of (1.2) satisfying

$$\begin{cases} P^k u \in C((-T, T), H^s) \cap X_b^s, \\ \displaystyle\sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|P^k u\|_{X_b^s} < \infty, \end{cases}$$

where $P = 5t\partial_t + x\partial_x$ is the generator of dilation for the linear part of the equation of (1.2).

Moreover, this solution becomes real analytic in both time and space variables; that is, there exist the positive constants C and A_1 such that

$$(1.4)\qquad |\partial_t^m \partial_x^l u(t, x)| \le C A_1^{m+l} (m + l)!$$

holds for all $(t, x) \in (-T, 0) \cup (0, T) \times \mathbf{R}$ and $l, m = 0, 1, 2, \cdots$.

The proof of this theorem is given in Sections 2 and 3. The detailed version of this paper will be submitted for publication elsewhere [9].

Notations. Let $\mathcal{F}_x$ be the Fourier transform in the x variable, and let $\mathcal{F}_\xi^{-1}$ and $\mathcal{F}_{\tau,\xi}^{-1}$ be the Fourier inverse transform in the ξ and (τ, ξ) variables, respectively. The Riesz operator D_x and its fractional derivative $\langle D_x \rangle^s$ are defined by

$$D_x = \mathcal{F}_\xi^{-1} |\xi| \mathcal{F}_x \quad \text{and} \quad \langle D_x \rangle^s = \mathcal{F}_\xi^{-1} \langle \xi \rangle^s \mathcal{F}_x,$$

respectively, where $\langle \cdot \rangle = (1 + |\cdot|)$. Similarly, $\langle D_{t,x} \rangle^s$ is defined by

$$\langle D_{t,x} \rangle^s = \mathcal{F}_{\tau,\xi}^{-1} \langle |\tau| + |\xi| \rangle^s \mathcal{F}_{t,x}.$$

$[A, B]$ denotes the commutator relation of two operators given by $AB - BA$. $L_t^p L_x^q$ denotes the space $L^p(\mathbf{R}_t; L^q(\mathbf{R}_x))$ for $1 \le p, q \le \infty$ with the norm

$$\|f\|_{L_t^p L_x^q} = \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(t, x)|^q dx \right)^{p/q} dt \right)^{1/p}.$$

We use the Sobolev space with both time and space variables

$$H_{t,x}^s(\mathbf{R}^2) = \{u \in \mathcal{S}'(\mathbf{R}^2) : \langle D_{t,x} \rangle^s u \in L_t^2 L_x^2\},$$

with the norm $\|\cdot\|_{H_{t,x}^s(\mathbf{R}^2)} = \|\langle D_{t,x} \rangle^s \cdot\|_{L_t^2 L_x^2}$. Moreover, $L_t^2(\mathbf{R}; H_x^s)$ denotes the space $L^2(\mathbf{R}_t; H^s(\mathbf{R}_x))$ with the norm $\|\cdot\|_{L_t^2(\mathbf{R}; H_x^s)} = \|\langle D_x \rangle^s \cdot\|_{L_t^2 L_x^2}$. For the constant A_0 appearing in Theorem 1.1 we put

$$\mathcal{A}_{A_0}(X_b^s) = \Big\{ \boldsymbol{f} = (f_0, f_1, \cdots); f_k \in X_b^s \ (k = 0, 1, \cdots) \text{ and } |||\boldsymbol{f}|||_{\mathcal{A}_{A_0}(X_b^s)} \equiv \sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|f_k\|_{X_b^s} < \infty \Big\},$$

For simplicity we make use of the notation

$$\sum_{\boldsymbol{k}} = \sum_{k = k_1 + k_2 + k_3 + k_4}.$$

2. Existence and uniqueness. In this section we give the existence and uniqueness of the solution of (1.2). Let $u_k = P^k u$ and $\phi_k(x) = (x\partial_x)^k \phi(x)$, and we derive the equation which u_k and $\phi_k(x)$ satisfy. Since $[x\partial_x, \partial_x] = -\partial_x$, it follows that

$$(2.1)\quad (P + l)^k \partial_x = \partial_x (P + (l - 1))^k \quad k, l = 0, 1, 2, \cdots.$$

Using (2.1) and the following relations

$$[\partial_t - \partial_x^5, P] = 5(\partial_t - \partial_x^5)$$

and

$$(\partial_t - \partial_x^5) P^k = (P + 5)^k (\partial_t - \partial_x^5),$$

we have from (1.2)

$$(2.2)\quad \begin{cases} \partial_t u_k - \partial_x^5 u_k = B_k(u), \quad t, x \in \mathbf{R}, \\ \qquad\qquad\qquad\qquad k = 0, 1, 2, \cdots, \\ u_k(0, x) = \phi_k(x), \quad x \in \mathbf{R}, \end{cases}$$

where

$$B_k(u) = \partial_x (P + 4)^k (u^3) + \partial_x (P + 4)^k ((\partial_x u)^2).$$

Using the Leibniz rule and (2.1), we can see that

$$\begin{aligned} B_k(u) &= \partial_x \sum_{l=0}^{k} \binom{k}{l} 4^{k-l} P^l (u^3) \\ &\quad + \partial_x \sum_{l=0}^{k} \binom{k}{l} 3^{k-l} (P + 1)^l (\partial_x u)^2 \\ &= \sum_{\boldsymbol{k}} \frac{k! 4^{k_4}}{k_1! k_2! k_3! k_4!} \partial_x (u_{k_1} u_{k_2} u_{k_3}) \\ &\quad + \sum_{\boldsymbol{k}} \frac{k! 3^{k_4} (-1)^{k_3}}{k_1! k_2! k_3! k_4!} \partial_x ((\partial_x u_{k_1})(\partial_x u_{k_2})). \end{aligned}$$

We will show the existence and uniqueness of the solution of (2.2).

Proposition 2.1. *Let*

$$(2.3)\qquad s > -1/4, \ \text{and } b \in (1/2, 1/2 + \sigma),$$

where $\sigma = \min\{s/5 + 1/20, 3/16\}$. Then for any $\boldsymbol{\phi} \equiv (\phi_0, \phi_1, \cdots)$ such that $\phi_k \in H^s(\mathbf{R})$ $(k = 0, 1, \cdots)$ and

$$\text{(2.4)} \qquad |||\boldsymbol{\phi}|||_{\mathcal{A}_{A_0}(H^s)} < \infty,$$

there exist a constant $T = T(\phi.) > 0$ *and a unique solution* $u_k \in C((-T, T), H^s) \cap X_b^s$ *of* (2.2) *satisfying*

$$\text{(2.5)} \qquad |||\boldsymbol{u}|||_{\mathcal{A}_{A_0}(X_b^s)} < \infty, \ \boldsymbol{u} \equiv (u_0, u_1, \cdots).$$

Remark 2.1. The uniqueness of the solution of (2.2) yields $u_k = P^k u$ for $k = 0, 1, 2 \cdots$. Moreover u_0 becomes a solution of (1.2), the uniqueness of which also follows.

To prove this proposition we prepare three lemmas, which play an important role in applying the contraction principle to the following system of the integral equations:

$$\text{(2.6)} \qquad \psi(t)u_k = \psi(t)e^{t\partial_x^5}\phi_k + \psi(t)\int_0^t e^{(t-t')\partial_x^5}\psi_T(t')B_k(u)(t')dt',$$

where

$$e^{t\partial_x^5} f \equiv \mathcal{F}_\xi^{-1}(e^{i\xi^5 t}\hat{f}(\xi)),$$

$\psi(t)$ denotes a cut-off function in $C_0^\infty(\mathbf{R})$ satisfying

$$\psi(t) = \begin{cases} 1, & \text{if} \ \ |t| \le 1, \\ 0, & \text{if} \ \ |t| > 2, \end{cases}$$

and $\psi_T(t) = \psi(t/T)$.

Lemma 2.1. *Let* $0 < T < 1$ *and let*

$$s \in \mathbf{R},\ b \in (1/2, 1),\ a', a \in (0, 1/2)\ (a' < a).$$

Then

$$\text{(2.7)} \qquad \|\psi(t)e^{t\partial_x^5}\phi(x)\|_{X_b^s} \le C_{0,s,b}\|\phi\|_{H^s},$$

$$\text{(2.8)} \qquad \left\|\psi(t)\int_0^t e^{(t-t')\partial_x^5}h(t')dt'\right\|_{X_b^s} \le C_{1,s,b}\|h\|_{X_{b-1}^s},$$

$$\text{(2.9)} \qquad \|\psi_T h\|_{X_{-a}^s} \le C_{2,s,-a,-a'}T^{(a-a')/4(1-a')}\|h\|_{X_{-a'}^s},$$

where $C_{0,s,b}$, $C_{1,s,b}$ *and* $C_{2,s,-a,-a'}$ *are constants depending on* s, b, $-a$ *and* $-a'$.

Lemma 2.2. *Let*

$$s > -1/4,\ \text{and}\ b, b' \in (1/2, 1/2+\sigma)\ (b \le b'),$$

where $\sigma = \min\{s/5 + 1/20, 3/16\}$. *Then*

$$\text{(2.10)} \qquad \|\partial_x((\partial_x u)(\partial_x v))\|_{X_{b'-1}^s} \le C_{3,s,b,b'}\|u\|_{X_b^s}\|v\|_{X_b^s},$$

where $C_{3,s,b,b'}$ *is a constant depending on* s, b *and* b'.

Proof. We can prove by proceeding to estimate carefully the potential which appears in an expression of the Bourgain norm of $\partial_x((\partial_x u)(\partial_x v))$ via duality. We note that this estimate is given by dividing the domain of integration of the potential into 30 subregions. □

Lemma 2.3. *Let*

$$s > -1/4,\ \text{and}\ b, b' \in (1/2, 3/4)\ (b \le b').$$

Then

$$\text{(2.11)} \qquad \|\partial_x(uvw)\|_{X_{b'-1}^s} \le C_{4,s,b,b'}\|u\|_{X_b^s}\|v\|_{X_b^s}\|w\|_{X_b^s},$$

where $C_{4,s,b,b'}$ *is a constant depending on* s, b *and* b'.

Proof. This lemma is proved by improving Chen, Li, Miao and Wu's argument used in the case where $b = b' \in (1/2, 3/4)$ [3]. □

Proof of Proposition 2.1. We define

$$X_{M_0} = \{\boldsymbol{f} \in \mathcal{A}_{A_0}(X_b^s);\ |||\boldsymbol{f}|||_{\mathcal{A}_{A_0}(X_b^s)} \le 2C_0M_0\},$$

where $M_0 = |||\boldsymbol{\phi}|||_{\mathcal{A}_{A_0}(H^s)}$.

We define a map $\Phi : X_{M_0} \longrightarrow X_{M_0}$ by $\Phi(u) = (\Phi_0(u), \Phi_1(u), \cdots)$ and

$$\text{(2.12)} \qquad \Phi_k(u) = \psi(t)e^{t\partial_x^5}\phi_k + \psi(t)\int_0^t e^{(t-t')\partial_x^5}\psi_T(t')B_k(u)(t')dt'.$$

Let b' and T be positive constants satisfying $b < b' < 1/2 + \sigma$ and

$$\text{(2.13)} \quad T < \min\{1,\ (24C_0^2C_5e^{4A_0}M_0^2 + 8C_0C_6e^{4A_0}M_0)^{-1/\mu}\},$$

respectively, where $C_0 = C_{0,s,b}$,

$$C_5 = C_{1,s,b}C_{2,s,b-1,b'-1}C_{4,s,b,b'},$$
$$C_6 = C_{1,s,b}C_{2,s,b-1,b'-1}C_{3,s,b,b'}.$$

We now show that Φ is a contraction mapping from X_{M_0} to itself. According to Lemmas 2.1, 2.2 and 2.3, we have for $u \in \mathcal{A}_{A_0}(X_b^s)$

$$\|\Phi_k(u)\|_{X_b^s} \le C_0\|\phi_k\|_{H^s} + C_5T^\mu\sum_{\boldsymbol{k}}\frac{k!4^{k_4}}{k_1!k_2!k_3!k_4!}\|u_{k_1}\|_{X_b^s}\|u_{k_2}\|_{X_b^s}\|u_{k_3}\|_{X_b^s} + C_6T^\mu\sum_{\boldsymbol{k}}\frac{k!3^{k_4}}{k_1!k_2!k_3!k_4!}\|u_{k_1}\|_{X_b^s}\|u_{k_2}\|_{X_b^s},$$

for any $k \ge 0$. Here $\mu = (b'-b)/\{4(2-b')\} > 0$. By taking a sum over k, we have

$$|||\Phi(u)|||_{\mathcal{A}_{A_0}(X_b^s)} = \sum_{k=0}^\infty \frac{{A_0}^k}{k!}\|\Phi_k(u)\|_{X_b^s}$$
$$\le C_0\|\boldsymbol{\phi}\|_{\mathcal{A}_{A_0}(H^s)} + C_5T^\mu\sum_{k_4=0}^\infty\frac{(4A_0)^{k_4}}{k_4!}\sum_{k_1=0}^\infty \times \frac{A_0^{k_1}}{k_1!}\|u_{k_1}\|_{X_b^s}\sum_{k_2=0}^\infty\frac{A_0^{k_2}}{k_2!}\|u_{k_2}\|_{X_b^s}\sum_{k_3=0}^\infty\frac{A_0^{k_3}}{k_3!}\|u_{k_3}\|_{X_b^s}$$

$$+ C_6T^\mu \sum_{k_4=0}^{\infty} \frac{(3A_0)^{k_4}}{k_4!} \sum_{k_3=0}^{\infty} \frac{A_0^{k_3}}{k_3!} \sum_{k_1=0}^{\infty} \frac{A_0^{k_1}}{k_1!} \|u_{k_1}\|_{X_b^s} \times \sum_{k_2=0}^{\infty} \frac{A_0^{k_2}}{k_2!} \|u_{k_2}\|_{X_b^s}.$$

Since $\boldsymbol{u} \in X_{M_0}$, we have from (2.13)

$$\begin{aligned}|||\Phi(u)|||_{\mathcal{A}_{A_0}(X_b^s)} &\le C_0\|\boldsymbol{\phi}\|_{\mathcal{A}_{A_0}(H^s)} \\ &\quad + C_5e^{4A_0}T^\mu|||\boldsymbol{u}|||^3_{\mathcal{A}_{A_0}(X_b^s)} + C_6e^{4A_0}T^\mu|||\boldsymbol{u}|||^2_{\mathcal{A}_{A_0}(X_b^s)} \\ &\le C_0M_0 + 8C_0^3C_5e^{4A_0}T^\mu M_0^3 + 4C_0^2C_6e^{4A_0}T^\mu M_0^2 \\ &\le C_0M_0 + \frac{1}{2}C_0M_0 = \frac{3}{2}C_0M_0,\end{aligned}$$

which implies $\Phi(u) \in X_{M_0}$. Similarly, we have for u and $\tilde{u} \in \mathcal{A}_{A_0}(X_b^s)$

$$\begin{aligned}(2.14)\quad |||\Phi(u) - \Phi(\tilde{u})|||_{\mathcal{A}_{A_0}(X_b^s)} &\le C_5e^{4A_0}T^\mu(|||\boldsymbol{u}|||^2_{\mathcal{A}_{A_0}(X_b^s)} \\ &\quad + |||\boldsymbol{u}|||_{\mathcal{A}_{A_0}(X_b^s)}|||\tilde{\boldsymbol{u}}|||_{\mathcal{A}_{A_0}(X_b^s)} + |||\tilde{\boldsymbol{u}}|||^2_{\mathcal{A}_{A_0}(X_b^s)}) \\ &\quad \times |||\boldsymbol{u} - \tilde{\boldsymbol{u}}|||_{\mathcal{A}_{A_0}(X_b^s)} + C_6e^{4A_0}T^\mu(|||\boldsymbol{u}|||_{\mathcal{A}_{A_0}(X_b^s)} \\ &\quad + |||\tilde{\boldsymbol{u}}|||_{\mathcal{A}_{A_0}(X_b^s)})|||\boldsymbol{u} - \tilde{\boldsymbol{u}}|||_{\mathcal{A}_{A_0}(X_b^s)} \\ &\le (12C_0^2C_5e^{4A_0}M_0^2 + 4C_0C_6e^{4A_0}M_0)T^\mu \\ &\quad \times |||\boldsymbol{u} - \tilde{\boldsymbol{u}}|||_{\mathcal{A}_{A_0}(X_b^s)} \\ &\le \frac{1}{2}|||\boldsymbol{u} - \tilde{\boldsymbol{u}}|||_{\mathcal{A}_{A_0}(X_b^s)}.\end{aligned}$$

Thus the mapping Φ is contraction from X_{M_0} to itself. We obtain a unique fixed point $u_k \in X_b^s$ satisfying

$$u_k(t) = \psi(t)e^{t\partial_x^5}\phi_k + \psi(t)\int_0^t e^{(t-t')\partial_x^5}\psi_T(t')B_k(u)(t')dt'$$

on the time interval $[-T, T]$ and $k = 0, 1, 2, \cdots$.

Uniqueness of the solution is also shown by using Bekiranov, Ogawa, and Ponce's argument in [1]. This completes the proof. □

3. Analyticity of the solution. In this section we prove the analyticity of the solution $u \equiv u_0$ given in Proposition 2.1. The proof is established by Propositions 3.1 and 3.2. To prove these propositions we prepare three lemmas.

Lemma 3.1. *Let (t_0, x_0) be an arbitrary fixed point in $(-T, 0) \cup (0, T) \times \mathbf{R}$.*

(1) Suppose that $b \in (0, 1]$, $r \in (-\infty, 0]$. Then for a sufficiently small $\varepsilon > 0$

$$(3.1)\quad \|\langle D_{t,x}\rangle^{5b}g\|_{L_t^2(\mathbf{R};H_x^r(\mathbf{R}))} \le K_{0,r,b}\{\|g\|_{X_{b-1}^r} + \|t\partial_x^5 g\|_{X_{b-1}^r} + \|P^5g\|_{X_{b-1}^r}\}$$

holds for all $g \in X_{b-1}^r$ satisfying

$$supp\, g \subset B_{2\varepsilon}(t_0, x_0),\ \ and\ t\partial_x^5 g, P^5 g \in X_{b-1}^r,$$

where $K_{0,r,b}$ is a constant depending on r and b.

(2) Let $\mu > 0$. Then for a sufficiently small $\varepsilon > 0$

$$(3.2)\quad \|\langle D_{t,x}\rangle^\mu g\|_{L_t^2L_x^2} \le K_{1,\mu}\{\|g\|_{H_{t,x}^{\mu-5}(\mathbf{R}^2)} + \|t\partial_x^5 g\|_{H_{t,x}^{\mu-5}(\mathbf{R}^2)} + \|P^5 g\|_{H_{t,x}^{\mu-5}(\mathbf{R})}\}$$

holds for all $g \in H_{t,x}^{\mu-5}(\mathbf{R}^2)$ satisfying

$$supp\, g \subset B_{2\varepsilon}(t_0, x_0),\ \ and\ t\partial_x^5 g, P^5 g \in H_{t,x}^{\mu-5}(\mathbf{R}^2),$$

where $K_{1,\mu}$ is a constant depending on μ.

Proof. We can prove these estimates by the localization argument and the L^2-boundedness theorem of some pseudo-differential operators. □

Lemma 3.2 [3]. *Let*

$$s > -7/4\ and\ b \in (1/2, 1/2 + \sigma)$$

where $\sigma = min\{1/4, (4s + 11)/8, (s + 6)/5\}$. Then

$$\|\partial_x(uv)\|_{X_{b-1}^s} \le K_{2,s,b}\|u\|_{X_b^s}\|v\|_{X_b^s},$$

where $K_{2,s,b}$ is a constant depending on s and b.

Let $\rho(t, x)$ be a smooth cut-off function around the freezing point (t_0, x_0) such that $\rho \in C_0^\infty(B_{2\varepsilon}(t_0, x_0))$.

Lemma 3.3. *Let $s, b \in \mathbf{R}$. Then*

$$\|\rho f\|_{X_b^s} \le K_{3,s,b,\rho}\varepsilon^{-|s|-9|b|}\|f\|_{X_b^{s+4|b|}}$$

holds for all $f(t, x) \in X_b^{s+4|b|}(\mathbf{R}^2)$, where

$K_{3,s,b,\rho} = K_{4,s,b}\|\langle\tau - \xi^5\rangle^{|b|}\langle\xi\rangle^{|s|+4|b|}\widehat{\rho}(\tau, \xi)\|_{L_\tau^1L_\xi^1}$ and $K_{4,s,b}$ is a constant depending on s and b.

Proof. We can prove by proceeding to estimate the potential which appears in an expression of the Bourgain norm of ρf via duality. □

Proposition 3.1. *Let $s > 1/8$ and let $b \in (1/2, 23/40)$. Then for a sufficiently small $\varepsilon > 0$, there exist positive constants K_5 and A_1 such that*

$$(3.3)\quad \|\rho P^k u\|_{H_{t,x}^{1/3}(\mathbf{R}^2)} + \|\rho P^k u\|_{L_t^2(\mathbf{R};H_x^1)} \le K_5A_1^k k!$$

holds for all $k = 0, 1, 2, \cdots$.

Proof. By Plancherel Theorem and Lemma 3.1 with $g = \rho P^k u$ we have

$$\begin{aligned}(3.4)\quad &\|\rho P^k u\|_{H_{t,x}^{1/3}(\mathbf{R}^2)} + \|\rho P^k u\|_{L_t^2(\mathbf{R};H_x^1)} \\ &\le \frac{2}{(2\pi)^2}\|\langle D_{t,x}\rangle^{5b}\rho P^k u\|_{L_t^2(\mathbf{R};H_x^r(\mathbf{R}))}, \\ &\le \frac{2}{(2\pi)^2}K_{0,r,b}\Big\{\|\rho P^k u\|_{X_{b-1}^r} \\ &\quad + \|t\partial_x^5(\rho P^k u)\|_{X_{b-1}^r} + \|P^5(\rho P^k u)\|_{X_{b-1}^r}\Big\},\end{aligned}$$

where

$$\begin{cases} r = s-2, & \text{if } 1/8 < s \le 2, \\ -15/8 < r \le 0, & \text{if } s > 2. \end{cases}$$

We note that $r \le s-2$ holds. Put $K_{6,s,b} = |||\boldsymbol{u}|||_{\mathcal{A}_{A_0}(X_b^s)}$. Since (2.5) and Remark 2.1 yield

$$(3.5) \quad \|P^k u\|_{X_b^s} \le K_{6,s,b}(A_0^{-1})^k k!, \quad k = 0,1,2,\cdots,$$

it follows from Lemma 3.3 that

$$(3.6) \quad \|\rho P^k u\|_{X_{b-1}^r} \le K_{3,r,b-1,\rho}\, \varepsilon^{-|r|-9|b-1|}\|P^k u\|_{X_{b-1}^s} \le K_{6,s,b-1}K_{3,r,b-1,\rho}\, \varepsilon^{-|r|-9|b-1|}(A_0^{-1})^k k!,$$

$$(3.7) \quad \|P^5(\rho P^k u)\|_{X_{b-1}^r} \le \sum_{l=0}^{5} \frac{5!}{(5-l)!l!}\|P^{5-l}\rho P^{l+k}u\|_{X_{b-1}^r} \le \sum_{l=0}^{5} \frac{5!}{(5-l)!l!} K_{3,r,b-1,\rho_l}\varepsilon^{-|r|-9|b-1|}\|P^{l+k}u\|_{X_{b-1}^s}$$
$$\le \max_{0\le l\le 5} K_{3,r,b-1,\rho_l}\, \varepsilon^{-|r|-9|b-1|} \times K_{6,s,b-1}\sum_{l=0}^{5}\frac{5!}{(5-l)!l!}\frac{(k+l)!}{2^k k!}(A_0^{-1})^l(2A_0^{-1})^k k! \le K_7 A_1^k k!,$$

where $\rho_l = P^{5-l}\rho$, K_7 is a some constant and $A_1 = (2A_0^{-1})$.

Now we estimate $\|t\partial_x^5(\rho P^k u)\|_{X_{b-1}^r}$. By using

$$(3.8) \quad t\partial_x^5(\rho P^k u) = t\rho(\partial_x^5 P^k u) + 5t\partial_x^2((\partial_x^2\rho)(\partial_x P^k u)) + 5t\partial_x((\partial_x\rho)(\partial_x^3 P^k u)) + t(\partial_x^5\rho)P^k u,$$

and

$$(3.9) \quad t(\partial_x^5 P^k u) = -\frac{1}{5}\{P^{k+1}u - x\partial_x P^k u\} + tB_k(u),$$

we have

$$(3.10) \quad \|t\partial_x^5(\rho P^k u)\|_{X_{b-1}^r} \le \frac{1}{5}\{\|\rho P^{k+1}u\|_{X_{b-1}^r} + \|\rho x\partial_x P^k u\|_{X_{b-1}^r}\} + \|t\rho B_k(u)\|_{X_{b-1}^r} + 5\|\partial_x^2(t(\partial_x^2\rho)(\partial_x P^k u))\|_{X_{b-1}^r} + 5\|\partial_x(t(\partial_x\rho)(\partial_x^3 P^k u))\|_{X_{b-1}^r} + \|t(\partial_x^5\rho)P^k u\|_{X_{b-1}^r}.$$

In the same manner as (3.6), we have

$$(3.11) \quad \|\rho P^{k+1}u\|_{X_{b-1}^r} \le K_{3,r,b-1,\rho}\varepsilon^{-|r|-9|b-1|}\|P^{k+1}u\|_{X_{b-1}^s} \le K_{3,r,b-1,\rho}\varepsilon^{-|r|-9|b-1|}K_{6,s,b-1}\frac{(k+1)!}{2^k k!}(A_0^{-1})(2A_0^{-1})^k k! \le K_8 A_1^k k!,$$

where K_8 is a some constant. By Lemmas 3.3 and 3.2, we have

$$(3.12) \quad \|\rho x\partial_x P^k u\|_{X_{b-1}^r} \le \|\partial_x(\rho x P^k u)\|_{X_{b-1}^r} + \|(\partial_x(\rho x))P^k u\|_{X_{b-1}^r}$$
$$\le K_{2,r,b}\|\rho x\|_{X_b^{s-2}}\|P^k u\|_{X_b^{s-2}} + K_{3,r,b-1,\partial_x(\rho x)}\varepsilon^{-|r|-9|b-1|}\|P^k u\|_{X_{b-1}^s}$$
$$\le \{K_{2,r,b}\|\rho x\|_{X_b^{s-2}} + K_{3,r,b-1,\partial_x(\rho x)}\varepsilon^{-|r|-9|b-1|}\} \times K_{6,s,b}(A_0^{-1})^k k!.$$

By Lemmas 2.2, 2.3 and 3.3, we have

$$(3.13) \quad \|t\rho B_k(u)\|_{X_{b-1}^r} \le K_{3,r,b-1,t\rho}\, \varepsilon^{-|r|-9|b-1|}\times \left\{C_{4,s,b}\sum_{\boldsymbol{k}}\frac{k!4^{k_4}}{k_1!k_2!k_3!k_4!}\|P^{k_1}u\|_{X_b^s}\|P^{k_2}u\|_{X_b^s}\|P^{k_3}u\|_{X_b^s} + C_{3,s,b}\sum_{\boldsymbol{k}}\frac{k!3^{k_4}}{k_1!k_2!k_3!k_4!}\|P^{k_1}u\|_{X_b^s}\|P^{k_2}u\|_{X_b^s}\right\}$$
$$\le K_{3,r,b-1,t\rho}\, \varepsilon^{-|r|-9|b-1|}\left\{C_{4,s,b}K_{6,s,b}^3\sum_{\boldsymbol{k}}\frac{4^{k_4}}{k_4!}\times (A_0^{-1})^{k_1+k_2+k_3} + C_{3,s,b}K_{6,s,b}^2\sum_{\boldsymbol{k}}\frac{3^{k_4}}{k_3!k_4!}(A_0^{-1})^{k_1+k_2}\right\}k!$$
$$\le K_{3,r,b-1,t\rho}\, \varepsilon^{-|r|-9|b-1|}e^{4/A_0^{-1}}\left\{C_{4,s,b}K_{6,s,b}^3\frac{(k+1)}{2^k} + C_{3,s,b}K_{6,s,b}^2\frac{(k+1)(k+2)}{2^k}\right\}(2A_0^{-1})^k k! \le K_9 A_1^k k!,$$

where K_9 is a some constant. We also have

$$(3.14) \quad \|\partial_x^2(t(\partial_x^2\rho)(\partial_x P^k u))\|_{X_{b-1}^r} \le \|\partial_x(t(\partial_x^2\rho)(\partial_x P^k u))\|_{X_{b-1}^{s-1}} \le C_{3,s-1,b}\|t\partial_x\rho\|_{X_b^{s-1}}\|P^k u\|_{X_b^{s-1}} \le C_{3,s-1,b}K_{6,s-1,b}\|t\partial_x\rho\|_{X_b^{s-1}}(A_0^{-1})^k k!,$$

and

$$(3.15) \quad \|\partial_x(t(\partial_x\rho)(\partial_x^3 P^k u))\|_{X_{b-1}^r} \le C_{3,s-2,b}\|t\rho\|_{X_b^{s-2}}\|\partial_x^2 P^k u\|_{X_b^{s-2}} \le C_{3,s-2,b}\|t\rho\|_{X_b^{s-2}}\|P^k u\|_{X_b^s} \le C_{3,s-2,b}K_{6,s,b}\|t\rho\|_{X_b^{s-2}}(A_0^{-1})^k k!.$$

In the same manner as (3.6), we have

$$(3.16) \quad \|t(\partial_x^5\rho)P^k u\|_{X_{b-1}^r} \le K_{3,r,b-1,t\partial_x^5\rho}K_{6,s,b-1}(A_0^{-1})^k k!.$$

Hence

$$(3.17) \quad \|t\partial_x^5(\rho P^k u)\|_{X_{b-1}^r} \le K_{10}A_1^k k!,$$

where K_{10} is some constant. Putting

$$K_5 = \frac{1}{6\pi^2} K_{0,r,b} \times \max\{K_{6,s,b-1} K_{3,r,b-1,\rho}\ \varepsilon^{-|r|-9|b-1|}, K_7, K_{10}\},$$

we have (3.3). □

Proposition 3.2. *Under the same assumption as in Proposition* 3.1*, there exist positive constants* C_{j+5} *and* $A_j (j = 2, 3, 4)$ *depending on* (t_0, x_0) *and* ε *such that*

$$\|\rho_1 P^k u\|_{H^{11/2}_{t,x}(\mathbf{R})} \le C_7 A_2^k k!, \tag{3.18}$$

$$\sup_{t\in I_{t_0,\varepsilon}} \|(t^{1/5}\partial_x)^l P^k u\|_{H^1(I_{x_0,\varepsilon})} \le C_8 A_3^{k+l}(k+l)! \tag{3.19}$$

and

$$\sup_{t\in I_{t_0,\varepsilon}} \|\partial_t{}^j \partial_x{}^l u\|_{H^1(I_{x_0,\varepsilon})} \le C_9 A_4^{j+l}(j+l)! \tag{3.20}$$

hold for all $k, l, j = 0, 1, 2, \cdots$*, where*

$$I_{t_0,\varepsilon} = (t_0 - \varepsilon, t_0 + \varepsilon),\ I_{x_0,\varepsilon} = (x_0 - \varepsilon, x_0 + \varepsilon),$$

ρ_1 *is a smooth cut-off function such that* $\rho_1 \equiv 1$ *on* $I_{t_0,\varepsilon} \times I_{x_0,\varepsilon}$.

Proof. The inequality (3.18) can be shown by using (3.2) with $g = \rho_1 P^k u$ and the similar method to the proof of Proposition 3.1. By induction on l we can prove (3.19). Here, we use (3.18) and the following relation given by (3.9):

$$\begin{aligned}(t^{1/5}\partial_x)^l P^k u &= t^{(l-5)/5}\partial_x^{l-5}(t\partial_x^5 P^k u) \\ &= -\frac{1}{5} t^{(l-5)/5}\partial_x^{l-5}\{P^{k+1}u - x\partial_x P^k u\} \\ &\quad + (t^{1/5})^l \partial_x^{l-5} B_k(u),\end{aligned} \tag{3.21}$$

where $l \ge 5$.

Now we show (3.20). From (3.19) it follows that

$$\begin{aligned}\sup_{t\in I_{t_0,\varepsilon}} \|\partial_x^l P^k u\|_{H^1(I_{x_0,\varepsilon})} &\le C_8 A_5^l A_3^k (k+l)! \\ &\le C_8 A_5^{k+l}(k+l)!, \quad k, l = 0, 1, 2, \cdots,\end{aligned} \tag{3.22}$$

where $A_5 = \max\{A_3|t_0 - \varepsilon|^{-1/5}, A_3\}$. Hence, by induction on m we have

$$\begin{aligned}&\sup_{t\in I_{t_0,\varepsilon}} \|(x\partial_x)^m \partial_x^l P^k u\|_{H^1(I_{x_0,\varepsilon})} \\ &\quad \le C_8 A_5^{k+l+m} C_{10}^m (k+l+m)! \quad k, l, m = 0, 1, 2, \cdots,\end{aligned} \tag{3.23}$$

where C_{10} is a constant satisfying $C_{10} \ge (|x_0| + \varepsilon + 1)e^{-A_5 C_{10}}$. Since $t\partial_t = (P - x\partial_x)/5$ and

$$P^{n_1}\partial_x^{n_2} = \partial_x^{n_2}(P - n_2)^{n_1}, \quad (n_1, n_2 = 0, 1, 2, \cdots),$$

it follows from (3.23) that

$$\sup_{t\in I_{t_0,\varepsilon}} \|(t\partial_t)^m \partial_x^l u\|_{H^1(I_{x_0,\varepsilon})} \le C_8 A_6^{l+m}(l+m)! \tag{3.24}$$

for all $l, m = 0, 1, 2, \cdots$, where $A_6 = \max\{A_5, C_{10}, 1\}(2 + (A_5 \max\{1, C_{10}\})^{-1})$. By induction on j we can prove that (3.24) implies

$$\begin{aligned}&\sup_{t\in I_{t_0,\varepsilon}} \|(t\partial_t)^m \partial_t^j \partial_x^l u\|_{H^1(I_{x_0,\varepsilon})} \\ &\quad \le C_8 A_6^{j+m+l} C_{11}^j (j+m+l)!, \quad j, l, m = 0, 1, 2, \cdots,\end{aligned} \tag{3.25}$$

where $C_{11} \ge |t_0 - \varepsilon|^{-1} e^{-A_6}$. Choosing $m = 0$ and $C_9 = C_8$ and $A_4 = \max\{A_6 C_{11}, A_6\}$ in (3.25), we have (3.20). This completes the proof of Proposition 3.2. □

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