On the inviscid Proudman-Johnson equation

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Abstract: We show that certain qualitative properties of classical solutions to the inviscid Proudman-Johnson equation are preserved as long as these solutions exist. This enables us to give a simple blow-up criterion.

Key words: Proudman-Johnson equation; blow-up.

1. Introduction. The inviscid Proudman-Johnson equation [12]

$$\begin{cases} f_{txx} + f f_{xxx} = f_x f_{xx}, \\ f(0,x) = f^0(x). \end{cases} \tag{1.1}$$

is obtained from the incompressible Euler equations in two space dimensions,

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = -\dfrac{1}{\rho}\nabla p \\ \operatorname{div}\mathbf{u} = 0, \end{cases} \tag{1.2}$$

by the separation of space variables for the stream function

$$\psi(t,x,y) = y\, f(t,x), \tag{1.3}$$

giving the velocity vector

$$\mathbf{u} = (\psi_y, -\psi_x).$$

A major open problem in partial differential equations is the blow-up problem for the incompressible Euler equation [1, 9]: can singularities arise in finite time from smooth initial velocities? The physical importance of this problem is far greater than the blow-up problem for the Navier-Stokes equation, despite the prominence of the latter as a Clay Millenium Problem [7]. Due to the fact that equation (1.1) describes solutions to the incompressible Euler equations, the blow-up issue for (1.1) with spatially periodic solutions satisfying

$$f(t,0) = f(t,1) \quad \text{and} \quad f_x(t,0) = f_x(t,1) \tag{1.4}$$

at instant t, is an open problem of great current interest. In this context notice that if instead of the incompressible Euler equations (1.2) we consider the incompressible Navier-Stokes equations

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = -\dfrac{1}{\rho}\nabla p + \nu\Delta\mathbf{u} \\ \operatorname{div}\mathbf{u} = 0, \end{cases}$$

where $\nu > 0$ is the constant viscosity, the *Ansatz* (1.3) yields the viscid Proudman-Johnson equation

$$f_{txx} + f f_{xxx} - \nu\, f_{xxxx} = f_x f_{xx},$$

lacking blow-up solutions (see [3]).

The classical Beale-Kato-Majda [2, 8] blow-up criterion for (1.2), says that the time integral of the maximum magnitude of the vorticity

$$\int_0^T \sup_{x,y} |\Delta\psi(t,x,y)|\, dt$$

controls blow-up or its absence. However, (1.3) yields a vorticity

$$-\Delta\psi(t,x,y) = -\,y\, f_{xx}(t,x) \tag{1.5}$$

of infinite supremum norm for $(x,y) \in [0,1]\times\mathbf{R}$, unless we are in the uninteresting case $f_{xx} \equiv 0$.

Our aim is to introduce a class of smooth functions that is preserved by the flow (1.1) and for which a simple blow-up criterion can be given.

2. Blow-up scenario. For integers $s \geq 1$ we denote by H^s the Sobolev space of square-integrable functions $F : [0,1] \to \mathbf{R}$ with square-integrable distributional derivatives up to order s. Okamoto [10] proved local existence in time of solutions to (1.1):

Theorem 2.1. *For any $f_x^0 \in H^s$ $(s \geq 1)$ satisfying (1.4) at time $t = 0$, there exists $T > 0$ and a unique solution $f_x \in \mathcal{C}([0,T]; H^s)$ of (1.1) satisfying (1.4) for all $t \in [0,T]$, with initial data $f(0,\cdot) = f^0$.*

Using (1.1) we see that if $f_x^0 \in H^s$ with $s \geq 2$, then the solution $f_x \in \mathcal{C}^1([0,T]; H^{s-1})$. Notice that the invariance of (1.1) under the transformation

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$f(t,x) \mapsto -f(t,-x)$ in combination with the above result shows that odd initial data f^0, satisfying

$$f^0(x) = -f^0(-x), \quad x \in \mathbf{R},$$

remain spatially odd for as long as they exist.

Particular weak solutions to (1.1) that blow up in finite time have been found and investigated in Childress *et al.* [4] and Okamoto [10], but no smooth blow-up solutions could be given so far in the literature. Global existence for classical solutions to (1.2), captured in our framework if $s \geq 2$, is ensured as long as

$$\int_0^1 f_{xx}^2(t,x)\,dx$$

does not blow-up [10]. While this criterion involves the vorticity (1.5), being thus reminiscent of the classical Beale-Kato-Majda blow-up criterion for (1.2), it is possible to give a simpler criterion for odd data. To this end, let us define

$$M(t) := \sup_{x\in[0,1]} \{f_x(t,x)\}. \tag{2.1}$$

Proposition 2.2. *If the initial data $f^0 \in H^3$ is odd, then the corresponding solution to* (1.1) *blows up in finite time if and only if* $\limsup_{t\uparrow T^*} M(t) = \infty$ *for some* $T^* < \infty$.

Proof. Multiplying (1.1) by f_{xx}, an integration by parts shows that

$$\frac{d}{dt}\int_0^1 f_{xx}^2\,dx = 3\int_0^1 f_x f_{xx}^2\,dx \leq 3M(t)\int_0^1 f_{xx}^2\,dx.$$

Gronwall's inequality [9] shows now that a bound on $M(t)$ provides us with a bound on $\int_0^1 f_{xx}^2\,dx$. □

Let us now introduce the class $\mathcal{F}$ of odd functions $f \in H^3$ with

$$\sup_{x\in[0,1]} \{f_x(x)\} = f_x(0). \tag{2.2}$$

For initial data $f^0 \in \mathcal{F}$ the above blow-up criterion simplifies. To show this we will use an abstract lemma by Constantin and Escher [5, 6]:

Lemma 2.3. *For* $f_x \in \mathcal{C}^1([0,T];\ H^1)$ *define the function M by* (2.1). *Then for every* $t \in [0,T]$, *there exists at least one point* $\xi(t) \in [0,1]$ *with* $M(t) = f_x(t,\xi(t))$, *and the function M is almost everywhere differentiable on* $(0,T)$ *with*

$$M'(t) = f_{xt}(t,\xi(t)) \quad \textit{a.e. on } (0,T).$$

With this lemma at hand, we can give a blow-up criterion for solutions to (1.1).

Theorem 2.4. *If the initial data $f^0 \in \mathcal{F}$, then the corresponding solution to* (1.1) *blows up in finite time if and only if* $\limsup_{t\uparrow T^*} f_x(t,0) = \infty$ *for some* $T^* < \infty$.

Proof. Integrating (1.1) once with respect to the spatial variable, we obtain

$$\partial_x(f_{tx} + ff_{xx} - f_x^2) = 0.$$

Using (1.4) we get

$$f_{tx} + ff_{xx} = f_x^2 - 2\int_0^1 f_x^2\,dx. \tag{2.3}$$

which, by Lemma 2.3, entails the ordinary differential equation

$$M'(t) = M^2(t) - 2\int_0^1 f_x^2\,dx \quad \text{a.e.} \tag{2.4}$$

Since f is odd and $f_x^0(0) = M(0)$ as $f^0 \in \mathcal{F}$, denoting

$$c(t) = \int_0^1 f_x^2\,dx,$$

we see that both functions $M(t)$ and $f_x(t,0)$ satisfy the ordinary differential equation $z'(t) = z^2(t) - 2c(t)$ with identical initial data. Thus $M(t) = f_x(t,0)$ for all times and we conclude by Proposition 2.2. □

We now introduce an interesting subfamily $\mathcal{F}^*$ of $\mathcal{F}$ by considering odd functions $f \in H^3$ such that f is convex on $(-1/2,0)$ and concave on $(0,1/2)$. Notice that if $f \in \mathcal{F}^*$ then $\int_{-1/2}^{1/2} f_x\,dx = 0$, and f_x is even and monotone on $(-1/2,0)$ and on $(0,1/2)$. We now show the relevance of $\mathcal{F}^*$ to (1.1).

Proposition 2.5. *If $f^0 \in \mathcal{F}^*$, then $f \in \mathcal{F}^*$ as long as the solution exists.*

Proof. Let $T^* > 0$ be the maximal existence time of the solution to (1.1) with initial data f^0. For $t \in [0,T^*)$ we define the diffeomorphism $\varphi(t,\cdot)$ of $[-1/2,1,2]$ as the solution to the system

$$\begin{cases} \varphi_t = f(t,\varphi), \\ \varphi(0,x) = x. \end{cases} \tag{2.5}$$

Since $f(t,0) = f(t,\pm 1/2) = 0$ as f is odd and satisfies (1.4), by uniqueness for the ordinary differential equation $z' = f(t,z)$ with initial data $z(0) = 0$, respectively $z(0) = \pm 1/2$ we infer from (2.5) that

$$\varphi(t,0) = 0,\ \varphi(t,\pm 1/2) = \pm 1/2, \tag{2.6}$$

for all $t \in [0,T^*)$. Define now

$$\theta(t,x) = f_{xx}(t,\varphi(t,x))$$

for $(t,x) \in [0,T^*) \times [-1/2,1/2]$. Using (2.5), we infer from (1.1) that $\theta_t = f_x(t,\varphi)\,\theta$. Thus

$$f_{xx}(t,\varphi(t,x)) = f_{xx}(0,x)\,e^{\int_0^t f_x(s,\varphi(s,x))\,ds}$$

for all $(t,x) \in [0,T^*) \times [-1/2,1/2]$. Since $f^0 \in \mathcal{F}$, the last relation in combination with (2.6) shows that for any $t \in (0,T^*)$ the function $f(t,\cdot)$ is convex on $(-1/2,0)$ and concave on $(0,1/2)$. We already know that $f(t,\cdot)$ has to be odd. Thus $f(t,\cdot) \in \mathcal{F}^*$. □

It is of interest to point out that (2.3) can be written as

$$\partial_x \left(f_t + f f_x - 2\int_0^x f_x^2\,dx + 2x\,c(t) \right) = 0. \tag{2.7}$$

If $f^0 \in H^3$ is odd, $f(t,\cdot)$ is also odd so that $f_t(t,0) = f(t,0) = 0$. Evaluating the differentiated expression in (2.7) at $x=0$, we infer that for $f^0 \in H^3$ odd,

$$f_t + f f_x = 2\int_0^x f_x^2\,dx - 2x\int_0^1 f_x^2\,dx. \tag{2.8}$$

Seeking separable solutions of (2.8) of the form

$$f(t,x) = \frac{F(x)}{T-t}$$

with $T>0$ fixed, amounts to solving the time-independent equation

$$F + F F_x = 2\int_0^x F_x^2\,dx - 2x\int_0^1 F_x^2\,dx$$

and leads to the blow-up solutions from [4, 10].

Remark 2.6. Similarly one can consider the *generalized* Proudman-Johnson equation introduced in [10, 11]. Results of this type will be exhibited in a forthcoming paper.

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