Two rigidity theorems on manifolds with Bakry-Emery Ricci curvature

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Abstract: In this paper, we generalize the Cheng's maximal diameter theorem and Bishop volume comparison theorem to the manifold with the Bakry-Emery Ricci curvature. As their applications, we obtain some rigidity theorems on the warped product.

Key words: Maximal diameter; Bakry-Emery Ricci curvature; volume comparison; warped product.

1. Introduction. The classical Myers's theorem [7] states that if (M, g) is a complete, connected Riemannian manifold of dimension n (≥ 2) such that $Ric \geq (n-1)kg > 0$, then its diameter $D = D(M)$ is less than or equal to $\frac{\pi}{\sqrt{k}}$. In particular, M is compact. It is natural to ask what happens if the diameter attains its maximal value. In [4] S.Y. Cheng proved, i.e. if (M, g) is a complete Riemannian manifold with $Ric \geq (n-1)kg > 0$, $D = \frac{\pi}{\sqrt{k}}$, then (M, g) is isometric to the sphere S_k^n.

In [1], Bakry and Ledoux proved an analogue of Myers's theorem and also provided a new analytic proof of Cheng's theorem based on Sobolev inequalities. Bakry and Ledoux's result implies that if (M, g) is a complete, connected Riemannian manifold of dimension n (≥ 2), assume that there is a smooth function h, $m \geq n$ and $k > 0$, such that the Bakry-Emery Ricci curvature $\widetilde{Ric} = Ric - \nabla\nabla h - \frac{1}{m-n}\nabla h \otimes \nabla h \geq (m-1)kg$. Then M is compact and $D \leq \frac{\pi}{\sqrt{k}}$. This result was also proved by Qian in [9] independently. At the end of the paper [1], the authors asked the rigidity question under the Bakry-Emery Ricci curvature setting. In this paper, the author wants to answer this question and prove the following rigidity theorem. Our method is motivated by Peterson [8].

Theorem 1.1. *Let (M, g) be a complete, connected Riemannian manifold of dimension n (≥ 2), assume that the Bakry-Emery Ricci curvature $\widetilde{Ric} = Ric - \nabla\nabla h - \frac{1}{m-n}\nabla h \otimes \nabla h \geq (m-1)kg > 0$, $m \geq n$, and $D = \frac{\pi}{\sqrt{k}}$, then (M, g) is isometric to the sphere S_k^n; moreover $h(x) = (m-n)\ln\frac{\sin(\sqrt{k}r)}{\sqrt{k}}$, where r is a distance function defined on S_k^n.*

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Remark 1.1. *From Theorem 1.1, we can see that when $m = n$, $h \equiv 0$ and $\widetilde{Ric} = Ric$, thus we reduce to Cheng's maximal diameter theorem.*

The rigidity theorems asserts that if a certain geometric quality is as large as possible relative to the pertinent lower bound on Ricci curvature, then the metric on the manifold in question is a warped product metric of a particular type. The notion of the warped product manifold was introduced by Bishop and O'Neil [2], where the authors used the warped product metric to construct a manifold of negative curvature.

Let $B = (B^l, g_B)$ and $F = (F^k, g_F)$ be two Riemannian manifolds with the dimension l and k. We denote by π and σ the projections of $B \times F$ onto B and F, respectively. For a nonnegative smooth function f defined on B, the warped product $N = B \times_f F$ is the product $N = B \times F$ furnished with the metric tensor g defined by $g = \pi^* g_B + f^2 \sigma^* g_F$, where $*$ denotes the pull back. The function f is referred to as the warping function. In [6], J. Lott pointed out that the Bakry-Emery Ricci curvature is in fact the horizontal part of the Ricci curvature of some warped product manifolds. Thus from Theorem 1.1 we obtain the following rigidity theorem on a warped product.

Corollary 1.1. *Let (M, g) be a complete, connected Riemannian manifold of dimension n (≥ 2), assume that the Bakry-Emery Ricci curvature $\widetilde{Ric} \geq (m-1)kg > 0$, $m \geq n$, and its diameter $D = \frac{\pi}{\sqrt{k}}$. If $N = M^n \times_{e^{\frac{h}{m-n}}} S_k^{m-n}$, then $N = S_k^n \times_{\left(\frac{1}{\sqrt{k}}\sin\sqrt{k}r\right)} S_k^{m-n}$, where r is a distance function defined on S_k^n.*

Now we want to discuss the rigidity theorem about the Bishop volume comparison. So far we only know the relative volume comparison theorem on

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manifolds with the Bakry-Emery Ricci curvature, see [3] (also see [5] or [10]). The problem about the Bishop volume comparison theorem remains open. We appreciate Prof. Bakry for his suggestion of studying this problem.

Let $B(p,r)$ be a ball centered p with a radius r in the manifold M, the weighted volume of $B(p,r)$ denotes by $vol_h(B(p,r)) = \int_{B(p,r)} dvol_h = \int_{B(p,r)} e^h dvol_g$ and $v(m,k,r)$ denotes the volume of a ball $B(k,r)$ of radius r in the space form M_k^m with a constant curvature k. Suppose that $Sn_k(r) = \frac{\sin\sqrt{k}r}{\sqrt{k}}$, $k>0$; $Sn_k(r) = r$, $k=0$; $Sn_k(r) = \frac{\sinh\sqrt{k}r}{\sqrt{k}}$, $k<0$; then we can prove the following Bishop volume comparison theorem.

Theorem 1.2. *Let (M,g) be a complete, connected Riemannian manifold of dimension n (≥ 2), assume that there is a function h satisfying $\lim_{r\to 0}\frac{e^h}{r^{m-n}} = 1$ such that the Bakry-Emery Ricci curvature $\widetilde{Ric} \geq (m-1)kg$, $m \geq n$, then*

$$vol_h(B(p,r)) \leq \frac{\omega_{n-1}}{\omega_{m-1}} v(m,k,r)$$

where ω_{n-1} is the volume of the n dimensional unit sphere; the equality holds if and only if $B(p,r)$ is isometric to $B(k,r)$; moreover $h(x) = (m-n)\ln Sn_k(r)$, where r is a distance function defined on $B(k,r)$.

Remark 1.2. *The function of $\frac{vol_h(B(p,r))}{v(m,k,r)}$ is a nonincreasing function. This is a well-known fact for experts. However people did not know what is its limits. So they could not obtain the Bishop volume comparison theorem. Our contribution of this paper is to find a warping function, which has a singularity at origin. Thus the initial condition of the warping function h is necessary. For example, $M = R^2$, $n=2$, $m=3$, $h(x)=1$, then $\widetilde{Ric} = 0$. The function of $\frac{vol_h(B(p,r))}{v(m,k,r)}$ is always a nonincreasing function, however its limit does not exist.*

As we know that

$$v(m,k,r) = \omega_{m-1}\int_0^r Sn_k^{m-1}(\rho)d\rho,$$

then we obtain the following interesting result, since we allow the number m to take any real number (not only integer).

Corollary 1.2. *Let (M,g) be a complete, connected Riemannian manifold of dimension n (≥ 2), assume that there is a function h satisfying $\lim_{r\to 0}\frac{e^{h(x)}}{r^{m-n}} = 1$ such that the Bakry-Emery Ricci curvature $\widetilde{Ric} \geq (m-1)kg$, $m \geq n$, then*

$$vol_h(B(p,r)) \leq \omega_{n-1}\int_0^r Sn_k^{m-1}(\rho)d\rho.$$

In particular, when $\widetilde{Ric} \geq 0$, the weighted volume has a sharp upper bound:

$$vol_h(B(p,r)) \leq \omega_{n-1}r^m.$$

As same as Corollary 1.1, we can obtain the following rigidity theorem on the warped product.

Corollary 1.3. *Let (M,g) be a complete, connected Riemannian manifold of dimension n (≥ 2), assume that there is a function h satisfying $\lim_{r\to 0}\frac{e^{h(x)}}{r^{m-n}} = 1$ such that the Bakry-Emery Ricci curvature $\widetilde{Ric} \geq (m-1)kg$, $m \geq n$, and $vol_h(B(p,r)) = \frac{\omega_{n-1}}{\omega_{m-1}}v(m,k,r)$. If $N = M^n \times_{e^{\frac{h}{m-n}}} S_k^{m-n}$, then $N = M_k^n \times_{Sn_k(r)} S_k^{m-n}$, where M_k^m is a space form with a constant curvature k, r is a distance function defined on M_k^n.*

2. Proof of Theorem 1.1 and Theorem 1.2. Firstly we introduce some lemmas. The following weighted Laplacian comparison theorem was proven by Qian [9] (also see [3] and [5]).

Lemma 2.1. (*Qian*) *Let (M,g) be a complete, connected Riemannian manifold of dimension n (≥ 2), assume that there is a smooth function h, $m \geq n$, such that Bakry-Emery Ricci curvature $\widetilde{Ric} \geq (m-1)kg$, then*

$$Lr(x) \leq (m-1)\frac{Sn_k'}{Sn_k}, \quad \forall x \in M \setminus cut(p),$$

where $L = \triangle + \nabla h \cdot \nabla$, $r(x)$ is a distance function from a fix point p, $cut(p)$ denotes the cut locus of the Riemannian manifold M with respect to the point p.

The following weighted volume comparison theorem can be found in [9] (also see [11]). However the limit of the relative volume comparison is a new result.

Lemma 2.2. *Let (M,g) be a complete, connected Riemannian manifold of dimension n with $\widetilde{Ric} \geq (m-1)kg$, $m \geq n$, then*

$$r \to \frac{vol_h(B(p,r))}{v(m,k,r)}$$

is a nonincreasing function; if the initial condition of the function h satisfying $\lim_{r\to 0}\frac{e^{h(x)}}{r^{m-n}} = 1$, then

$$\lim_{r\to 0}\frac{vol_h(B(p,r))}{v(m,k,r)} = \frac{\omega_{n-1}}{\omega_{m-1}}.$$

Secondly we want to use the above Lemmas to prove Theorem 1.1:

Proof. Fix $p,q\in M$ such that $d(p,q)=\frac{\pi}{\sqrt{k}}$. Define $r(x)=d(p,x)$, $\widetilde{r}(x)=d(q,x)$. Firstly, we want to claim the following fact:

$$r(x)+\widetilde{r}(x)=d(p,q)=\frac{\pi}{\sqrt{k}},\quad \forall x\in M. \tag{2.1}$$

The above fact also tell us that the function r and $\widetilde{r}$ is smooth on $M-\{p,q\}$. Now we suppose that (2.1) does not hold, from the triangle inequality we know that

$$d(p,x)+d(q,x)>\frac{\pi}{\sqrt{k}}.$$

So we can find $\varepsilon>0$ such that

$$d(p,x)+d(q,x)=\frac{\pi}{\sqrt{k}}+2\varepsilon=d(p,q)+2\varepsilon.$$

Then the balls $B(p,r_1)$ and $B(q,r_2)$ and $B(x,\varepsilon)$ are pairwise disjoint, when $r_1\le d(p,x)$, $r_2\le d(q,x)$ and $r_1+r_2=\frac{\pi}{\sqrt{k}}$. Thus by Lemma 2.2, we have that

$$\begin{aligned}1&=\frac{vol_h(M)}{vol_h(M)}\\&\ge\frac{vol_h(B(x,\varepsilon))+vol_h(B(p,r_1))+vol_h(B(q,r_2))}{vol_h(M)}\\&\ge\frac{v(m,k,\varepsilon)+v(m,k,r_1)+v(m,k,r_2)}{v(m,k,\frac{\pi}{\sqrt{k}})}\\&=\frac{v(m,k,\varepsilon)}{v(m,k,\frac{\pi}{\sqrt{k}})}+1,\end{aligned}$$

which is a contradiction.

Secondly, we can use (2.1) and Lemma 2.1 to prove the following fact:

$$Hess\ r=\frac{Sn_k'(r)}{Sn_k(r)}ds_{n-1}^2. \tag{2.2}$$

In fact, from (2.1) we know that $Lr=-L\widetilde{r}$. On the other hand, from Lemma 2.1 we have that

$$\begin{aligned}(m-1)\frac{Sn_k'(r)}{Sn_k(r)}\ge Lr=-L\widetilde{r}\ge-(m-1)\frac{Sn_k'(\widetilde{r})}{Sn_k(\widetilde{r})}\\=(m-1)\frac{Sn_k'(r)}{Sn_k(r)}.\end{aligned}$$

So we obtain that

$$Lr=(m-1)\frac{Sn_k'(r)}{Sn_k(r)}. \tag{2.3}$$

By (2.3) and the fact of $\partial_r\triangle r+|Hess\ r|^2=-Ric(\partial r,\partial r)$, we see that

$$\begin{aligned}-(m-1)k&=\partial_rLr+\frac{1}{m-1}(Lr)^2\\&=\partial_r\triangle r+\partial_r^2h+\frac{1}{m-1}(\triangle r+\partial_rh)^2\\&\le\partial_r\triangle r+\partial_r^2h+\frac{1}{n-1}(\triangle r)^2+\frac{1}{m-n}(\partial_rh)^2\\&\le\partial_r\triangle r+\partial_r^2h+|Hess\ r|^2+\frac{1}{m-n}(\partial_rh)^2\\&=-\widetilde{Ric}(\partial r,\partial r)\\&\le-(m-1)k.\end{aligned}$$

So the equalities in the above inequalities always hold, i.e.

$$\frac{1}{m-1}(Lr)^2=\frac{1}{n-1}(\triangle r)^2+\frac{1}{m-n}(\partial_rh)^2,$$

$$|Hess\ r|^2=\frac{1}{n-1}(\triangle r)^2.$$

Thus we have that

$$\frac{\triangle r}{n-1}=\frac{\partial_rh}{m-n}=\frac{Lr}{m-1}, \tag{2.4}$$

and

$$\begin{aligned}Hess\ r&=\frac{\triangle r}{n-1}g_r=\frac{Lr}{m-1}g_r\\&=\frac{Sn_k'(r)}{Sn_k(r)}ds_{n-1}^2.\end{aligned} \tag{2.5}$$

Let X_i, $i=1,\dots,n-1$, be the orthonormal eigenvectors of $Hess\ r$ at r, then from (2.5), we have

$$\nabla_{X_i}\frac{\partial}{\partial r}=\sqrt{k}\cot\sqrt{k}rX_i.$$

Extend X_i in such a way that $[X_i,\frac{\partial}{\partial r}]=0$ at r, thus we can compute the sectional curvature of M.

$$\begin{aligned}&Sec\left(X_i,\frac{\partial}{\partial r}\right)\\&=-<\nabla_{\frac{\partial}{\partial r}}\nabla_{X_i}\frac{\partial}{\partial r},X_i>\\&=-<\nabla_{\frac{\partial}{\partial r}}(\sqrt{k}\cot\sqrt{k}r)X_i,X_i>\\&=k\csc^2\sqrt{k}r-\sqrt{k}\cot\sqrt{k}r<\nabla_{\frac{\partial}{\partial r}}X_i,X_i>\\&=k\csc^2\sqrt{k}r-\sqrt{k}\cot\sqrt{k}r<\nabla_{X_i}\frac{\partial}{\partial r},X_i>\\&=k\csc^2\sqrt{k}r-(\sqrt{k}\cot\sqrt{k}r)^2=k.\end{aligned}$$

By (2.3) and (2.4), we obtain that

$$h(x) = (m-n)\ln\frac{\sin(\sqrt{k}r)}{\sqrt{k}} + C.$$

Since the function h in the Bakry-Emery Ricci curvature is a translation invariant, constant C has to be zero, i.e. $h(x) = (m-n)\ln\frac{\sin(\sqrt{k}r)}{\sqrt{k}}$. □

Finally we prove Theorem 1.2. By Lemma 2.2, we know that $vol_h(B(p,r)) \leq \frac{\omega_{n-1}}{\omega_{m-1}} v(m,k,r)$. So we only prove the rigidity part of Theorem 1.2.

Proof. Set $dvol_h = e^h dvol_g = \lambda_h(r,\theta)dr\wedge d\theta$, $dvol_k = \lambda_k(r)dr\wedge d\theta$, where $\lambda_k(r) = Sn_k^{m-1}(r)$. From Lemma 2.1, we know that

$$Lr = \partial_r(\ln\lambda_h(r,\theta)) \leq \partial_r(\ln\lambda_k(r)). \tag{2.6}$$

Setting $F(r) = \frac{\lambda_h(r,\theta)}{\lambda_k(r)}$, then we compute the derivative of $F(r)$.

$$F'(r) = \frac{\partial_r\lambda_h(r,\theta)\lambda_k(r) - \lambda_h(r,\theta)\partial_r\lambda_k(r)}{\lambda_k^2(r)} = \frac{\lambda_h(r,\theta)}{\lambda_k(r)}(\partial_r\ln\lambda_h(r,\theta) - \partial_r\ln\lambda_k(r)).$$

By (2.6), we know that the function $F(r)$ is a non-increasing function. On the other hand, by the initial condition of the function h, let the distance function r tend to zero, we can easily obtain that

$$\lambda_h(r,\theta) \leq \lambda_k(r). \tag{2.7}$$

Since

$$\int_0^r\int_{S^{n-1}}\lambda_h(\rho,\theta)d\rho\wedge d\theta = \frac{\omega_{n-1}}{\omega_{m-1}}\int_0^r\int_{S^{m-1}}\lambda_k(\rho)d\rho\wedge d\theta$$

Taking the derivative of the both sides of the above equation, then we get that

$$\int_{S^{n-1}}\lambda_h(r,\theta)d\theta = \frac{\omega_{n-1}}{\omega_{m-1}}\int_{S^{m-1}}\lambda_k(r)d\theta = \int_{S^{n-1}}\lambda_k(r)d\theta$$

By (2.7), we know that

$$\lambda_h(r,\theta) = \lambda_k(r).$$

Thus

$$Lr = (m-1)\frac{Sn_k'(r)}{Sn_k(r)}.$$

With the same Proof of Theorem 1.1, we can easily prove Theorem 1.2. □

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References

[1] D. Bakry and M. Ledoux, Sobolev inequalities and Myers's diameter theorem for an abstract Markov generator, Duke Math. J. **85** (1996), no. 1, 253–270.

[2] R. L. Bishop and B. O'Neill, Manifolds of negative curvature, Trans. Amer. Math. Soc. **145** (1969), 1–49.

[3] D. Bakry and Z. Qian, Volume comparison theorems without Jacobi fields, in *Current trends in potential theory*, pp. 115–122, Theta, Bucharest.

[4] S. Y. Cheng, Eigenvalue comparison theorems and its geometric applications, Math. Z. **143** (1975), no. 3, 289–297.

[5] X.-D. Li, Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds, J. Math. Pures Appl. (9) **84** (2005), no. 10, 1295–1361.

[6] J. Lott, Some geometric properties of the Bakry-Emery-Ricci tensor, Comment. Math. Helv. **78** (2003), no. 4, 865–883.

[7] S. B. Myers, Connections between differential geometry and topology. I. Simply connected surfaces, Duke Math. J. **1** (1935), no. 3, 376–391.

[8] P. Petersen, *Riemannian geometry*, Springer, New York, 1998.

[9] Z. Qian, Estimates for weighted volumes and applications, Quart. J. Math. Oxford Ser. (2) **48** (1997), no. 190, 235–242.

[10] G. Wei and W. Wylie, Comparison Geometry for the Bakry-Emery Ricci Tensor, arxiv:math.DG/0706.1120.

[11] G. Wei and W. Wylie, Comparison Geometry for the Smooth Metric Measure Spaces, in *4th International Congress of Chinese Mathematicians* (*Hangzhou, China, 2007*), Proceedings of the 4th International Congress of Chinese Mathematicians, Hangzhou, China, 2007, Vol. II 191-202.