A mandala of Legendrian dualities for pseudo-spheres in semi-Euclidean space

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(Communicated by Heisuke HIRONAKA, M.J.A., March 12, 2009)

Abstract: We show six Legendrian dualities between pseudo-spheres in semi-Euclidean space which are basic tools for the study of extrinsic differential geometry of submanifolds in these pseudo-spheres.

Key words: Legendrian dualities; semi-Euclidean space; pseudo-spheres.

1. Introduction. A theorem of Legendrian dualities for pseudo-spheres in Minkowski space has been shown by the second author in [8] which is now a fundamental tool for the study of extrinsic differential geometry on submanifolds in these pseudo-spheres from the view point of Singularity theory (cf., [8,11,12]). In this paper we consider similar Legedrian dualities between pseudo-spheres in general semi-Euclidean space. The main results (cf., Theorems 3.1 and 3.2) are simple generalizations of the previous results in [8,10]. However, there are some new applications and information.

The Lorentzian space form with negative sectional curvature is called Anti de Sitter space which is given as a pseudo-sphere with a negative radius in semi-Euclidean space with index 2. This space is a very important subject in Physics (the theory of general relativity, the string theory and the brane world scenario etc. [19–21]). We can apply the Legendrian duality theorem to this space and obtain some new geometric properties of submanifolds. The detailed arguments on this application will be appeared in elsewhere.

Recently there appeared several results on submanifolds in hyperbolic space and de Sitter space which are pseudo-spheres in Minkowski space [1,6,7,17,18]. We give an interesting interpretation on the set of Legendrian dualities from a new point of view (i.e., a mandala of Legendrian dualities in §3). We can add some new information on the above results from this point of view.

2. Basic notions. In this section we prepare basic notions on semi-Euclidean space. Let $\mathbf{R}^{n+1} = \{(x_1, \dots, x_{n+1}) \mid x_i \in \mathbf{R}, i = 1, \dots, n+1\}$ be an $(n+1)$-dimensional vector space. For any vectors $\boldsymbol{x} = (x_1, \dots, x_{n+1})$, $\boldsymbol{y} = (y_1, \dots, y_{n+1})$ in $\mathbf{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -\sum_{i=1}^{r} x_i y_i + \sum_{i=r+1}^{n+1} x_i y_i$. The space $(\mathbf{R}^{n+1}, \langle , \rangle)$ is called *semi-Euclidean* $(n+1)$*-space with index* r and denoted by $\mathbf{R}^{n+1}_r$. We say that a vector $\boldsymbol{x}$ in $\mathbf{R}^{n+1}_r \setminus \{\mathbf{0}\}$ is *spacelike*, *null* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0 respectively. The norm of the vector $\boldsymbol{x} \in \mathbf{R}^{n+1}_r$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. We have the following three kinds of pseudo-spheres in $\mathbf{R}^{n+1}_r$: The *pseudohyperbolic n-space with idex* $r-1$ is defined by

$$H^n_{r-1} = \{\boldsymbol{x} \in \mathbf{R}^{n+1}_r \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},$$

the *pseudo n-sphere with idex* r by

$$S^n_r = \{\boldsymbol{x} \in \mathbf{R}^{n+1}_r \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$$

and the (*open*) *nullcone* by

$$\Lambda^n = \{\boldsymbol{x} \in \mathbf{R}^{n+1}_r \setminus \{\mathbf{0}\} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.$$

In relativity theory $\mathbf{R}^{n+1}_1$ is called *Minkoski* $(n+1)$*-space*, S^n_1 is *de Sitter n-space* and H^n_1 is *Anti de Sitter n-space* which is denoted by AdS^n. These are the Lorentzian space forms. Moreover, H^n_0 is called *hyperbolic n-space* and S^n_0 is the Euclidean unit sphere which are the Riemannian space forms.

3. Legendrian dualities. We now review some properties of contact manifolds and Legendrian submanifolds. Let N be a $(2n+1)$-dimensional smooth manifold and K be a tangent hyperplane field on N. Locally such a field is defined as

2000 Mathematics Subject Classification. Primary 53A35, 58C25.

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doi: 10.3792/pjaa.85.49

the field of zeros of a 1-form α. The tangent hyperplane field K is *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of N. We say that (N, K) is a *contact manifold* if K is a non-degenerate hyperplane field. In this case K is called a *contact structure* and α is a *contact form*. Let $\phi : N \longrightarrow N'$ be a diffeomorphism between contact manifolds (N, K) and (N', K'). We say that ϕ is a *contact diffeomorphism* if $d\phi(K) = K'$. Two contact manifolds (N, K) and (N', K') are *contact diffeomorphic* if there exists a contact diffeomorphism $\phi : N \longrightarrow N'$. A submanifold $i : L \subset N$ of a contact manifold (N, K) is said to be *Legendrian* if $\dim L = n$ and $di_x(T_xL) \subset K_{i(x)}$ at any $x \in L$. We say that a smooth fiber bundle $\pi : E \longrightarrow M$ is called a *Legendrian fibration* if its total space E is furnished with a contact structure and its fibers are Legendrian submanifolds. Let $\pi : E \longrightarrow M$ be a Legendrian fibration. For a Legendrian submanifold $i : L \subset E$, $\pi \circ i : L \longrightarrow M$ is called a *Legendrian map*. The image of the Legendrian map $\pi \circ i$ is called a *wavefront set* of i which is denoted by $W(L)$. For any $p \in E$, it is known that there is a local coordinate system $(x_1, \dots, x_m, p_1, \dots, p_m, z)$ around p such that

$$\pi(x_1, \dots, x_m, p_1, \dots, p_m, z) = (x_1, \dots, x_m, z)$$

and the contact structure is given by the 1-form

$$\alpha = dz - \sum_{i=1}^{m} p_i dx_i$$

(cf. [2], 20.3).

In [8] we have shown the basic duality theorem which is the fundamental tool for the study of spacelike hypersurfaces in Minkowski pseudo-spheres. In this paper we consider the similar dualities in semi-Euclidean space. We now consider the following four double fibrations:

(1) (a) $H^n_{r-1} \times S^n_r \supset \Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\}$,
(b) $\pi_{11} : \Delta_1 \longrightarrow H^n_{r-1}$, $\pi_{12} : \Delta_1 \longrightarrow S^n_r$,
(c) $\theta_{11} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_1$, $\theta_{12} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_1$.

(2) (a) $H^n_{r-1} \times \Lambda^n \supset \Delta_2 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -1\}$,
(b) $\pi_{21} : \Delta_2 \longrightarrow H^n_{r-1}$, $\pi_{22} : \Delta_2 \longrightarrow \Lambda^n$,
(c) $\theta_{21} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_2$, $\theta_{22} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_2$.

(3) (a) $\Lambda^n \times S^n_r \supset \Delta_3 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 1\}$,
(b) $\pi_{31} : \Delta_3 \longrightarrow \Lambda^n$, $\pi_{32} : \Delta_3 \longrightarrow S^n_r$,
(c) $\theta_{31} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_3$, $\theta_{32} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_3$.

(4) (a) $\Lambda^n \times \Lambda^n \supset \Delta_4 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -2\}$,
(b) $\pi_{41} : \Delta_4 \longrightarrow \Lambda^n$, $\pi_{42} : \Delta_4 \longrightarrow \Lambda^n$,
(c) $\theta_{41} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_4$, $\theta_{42} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_4$.

Here, $\pi_{i1}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}$, $\pi_{i2}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}$, $\langle d\boldsymbol{v}, \boldsymbol{w} \rangle = -w_0 dv_0 + \sum_{i=1}^n w_i dv_i$ and $\langle \boldsymbol{v}, d\boldsymbol{w} \rangle = -v_0 dw_0 + \sum_{i=1}^n v_i dw_i$ are one-forms on $\mathbf{R}^{n+1}_r \times \mathbf{R}^{n+1}_r$.

We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over Δ_i which is denoted by K_i. The basic duality theorem is the following theorem:

Theorem 3.1. *Under the same notations as the previous paragraph, each* (Δ_i, K_i) $(i = 1, 2, 3, 4)$ *is a contact manifold and both of* π_{ij} $(j = 1, 2)$ *are Legendrian fibrations. Moreover those contact manifolds are contact diffeomorphic each other.*

Before we give the proof, we include here a quick review on the canonical contact structure on the projective cotangent bundle over a manifold. Let $\pi : PT^*(M) \to M$ be the projective cotangent bundle over an n-dimensional manifold M. This fibration can be considered as a Legendrian fibration with the canonical contact structure K on $PT^*(M)$. We now review geometric properties of this space. Consider the tangent bundle $\tau : TPT^*(M) \to PT^*(M)$ and the differential map $d\pi : TPT^*(M) \to N$ of π. For any $X \in TPT^*(M)$, there exists an element $\alpha \in T^*(M)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x(M)$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(M)$ by

$$K = \{X \in TPT^*(M) \mid \tau(X)(d\pi(X)) = 0\}.$$

For a local coordinate neighborhood $(U, (x_1, \dots, x_n))$ on M, we have a trivialization $PT^*(U) \cong U \times P(\mathbf{R}^{n-1})^*$ and we call $((x_1, \dots, x_n), [\xi_1 : \dots : \xi_n])$ *homogeneous coordinates*, where $[\xi_1 : \dots : \xi_n]$ are homogeneous coordinates of the dual projective space $P(\mathbf{R}^{n-1})^*$. It is easy to show that $X \in K_{(x,[\xi])}$ if and only if $\sum_{i=1}^n \mu_i \xi_i = 0$, where $d\pi(X) = \sum_{i=1}^n \mu_i \frac{\partial}{\partial x_i}$. This means that the contact form α on the affine coordinates $U_j = \{(\boldsymbol{x}, [\xi]) \mid \xi_j \neq 0\} \subset PT^*(U)$ is given by $\alpha = \sum_{i=1}^n (\xi_i/\xi_j) dx_i$.

Proof. By definition we can easily show that each Δ_i $(i = 1, 2, 3, 4)$ is a smooth submanifold in $\mathbf{R}^{n+1}_r \times \mathbf{R}^{n+1}_r$ and each π_{ij} $(i = 1, 2, 3, 4; j = 1, 2)$ is a smooth fibration. It also follows from the definition of θ_{ij} that each fibre of π_{ij} is an integral submanifold of K_i $(i = 1, 2, 3, 4)$.

Firstly, we show that (Δ_1, K_1) is a contact manifold. For any $\boldsymbol{v} = (v_1, \cdots, v_{n+1}) \in H^n_{r-1}$, we have $\sum_{i=1}^r v_i^2 \neq 0$. Therefore $(v_1, \dots, v_r) \neq (0, \dots, 0)$. We consider a coordinate neighborhood $V_1^+ = \{\boldsymbol{v} = (v_1, \dots, v_{n+1}) \in H^n_{r-1} \mid v_1 > 0\}$ on which we have $v_1 = \sqrt{-\sum_{i=2}^{r} v_i^2 - \sum_{i=r+1}^{n+1} v_i^2 + 1}$. Therefore, we regard that $(v_2, \cdots, v_{n+1})$ is the local coordinates on V_1^+. For any $\boldsymbol{w} = (w_1, \ldots, w_{n+1}) \in S_r^n$, we have $\sum_{i=r+1}^{n+1} w_i^2 \neq 0$, so that $(w_{r+1}, \ldots, w_{n+1}) \neq (0, \ldots, 0)$. We also consider a coordinate neighborhood $W_{r+1}^+ = \{\boldsymbol{w} \in S_r^n \mid w_{r+1} > 0\}$. Then $V_1^+ \times W_{r+1}^+$ is one of the local coordinate of $H_{r-1}^n \times S_r^n$. We now define a mapping

$$\Phi : \Delta_1 \cap (V_1^+ \times W_{r+1}^+) \longrightarrow PT^*H_{r-1}^n \mid V_1^+$$

by

$$\begin{aligned}\Phi(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v}, [(w_1 v_2 - w_2 v_1) : \cdots : (w_1 v_r - w_r v_1) :\\ (-w_1 v_{r+1} + w_{r+1} v_1) : \cdots : (-w_1 v_{n+1} + w_{n+1} v_1)])\end{aligned}$$

Let $(v_2, \ldots, v_{n+1}, [\xi_2 : \cdots : \xi_{n+1}])$ be homogeneous coordinates of $PT^*H_{r-1}^n \mid V_1^+ \equiv V_1^+ \times P(\mathbf{R}^{n-1})^*$. We have the canonical contact form $\alpha = \sum_{i=2}^{n+1} (\xi_i/\xi_j) dv_i$ on $PT^*H_{r-1}^n$ over $V_1^+ \times U_j$, where $U_j = \{[\xi] \mid \xi_j \neq 0\}$. It follows that

$$\begin{aligned}\Phi^*\alpha &= \frac{\pm v_1}{w_j v_1 - w_1 v_j} \left(-\sum_{i=1}^{r} w_i dv_i + \sum_{i=r+1}^{n+1} w_i dv_i \right)\\ &= \frac{\pm v_1}{w_j v_1 - w_1 v_j} \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_1 = \frac{\pm v_1}{w_j v_1 - w_1 v_j}\, \theta_{11},\end{aligned}$$

where $\pm$ depends on j of $\Phi^{-1}(V_1^+ \times U_j)$. Since

$$\Delta_1 \cap (V_1^+ \times W_{r+1}^+) = \bigcup_{j=2}^{n+1} \Phi^{-1}(V_1^+ \times U_j),$$

θ_{11} is a contact form on $\Delta_1 \cap (V_1^+ \times W_{r+1}^+)$ such that Φ is a contact morphism. We also have the similar calculation as the above on the other coordinate neighborhoods. Thus $(\Delta_1, \theta_{11}^{-1}(0))$ is a contact manifold. For the other Δ_i $(i = 2, 3, 4)$ we define smooth mappings $\Psi_{1i} : \Delta_1 \longrightarrow \Delta_i$ by

$$\begin{aligned}\Psi_{12}(\boldsymbol{v}, \boldsymbol{w}) &= (\boldsymbol{v}, \boldsymbol{v} + \boldsymbol{w}),\\ \Psi_{13}(\boldsymbol{v}, \boldsymbol{w}) &= (\boldsymbol{v} - \boldsymbol{w}, -\boldsymbol{w}),\\ \Psi_{14}(\boldsymbol{v}, \boldsymbol{w}) &= (\boldsymbol{v} - \boldsymbol{w}, \boldsymbol{v} + \boldsymbol{w}).\end{aligned}$$

We can construct the converse mappings defined by

$$\begin{aligned}\Psi_{21}(\boldsymbol{v}, \boldsymbol{w}) &= (\boldsymbol{v}, \boldsymbol{w} - \boldsymbol{v}),\\ \Psi_{31}(\boldsymbol{v}, \boldsymbol{w}) &= (\boldsymbol{v} - \boldsymbol{w}, -\boldsymbol{w})\\ \Psi_{41}(\boldsymbol{v}, \boldsymbol{w}) &= \left(\frac{\boldsymbol{v} + \boldsymbol{w}}{2}, \frac{\boldsymbol{w} - \boldsymbol{v}}{2} \right).\end{aligned}$$

Therefore, Ψ_{1i} are diffeomorphisms. Moreover, we have

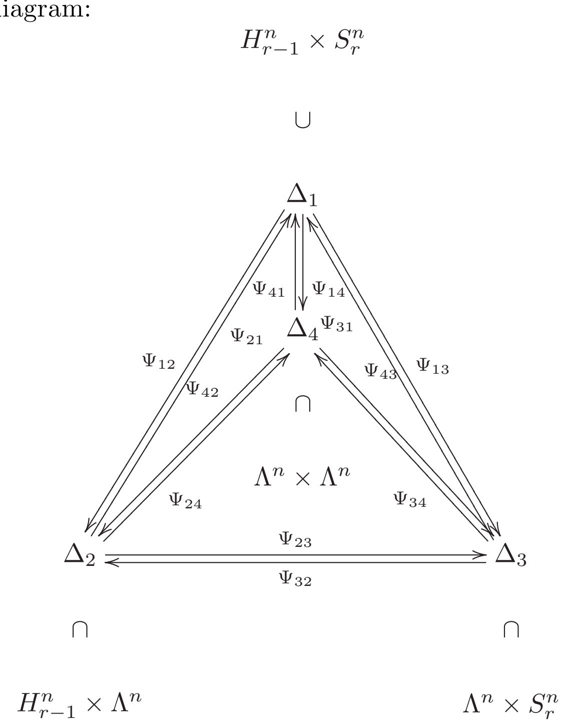
$$\begin{aligned}\Psi_{12}^*\theta_{21} &= \langle d\boldsymbol{v}, \boldsymbol{v} + \boldsymbol{w} \rangle | \Delta_1 = \langle d\boldsymbol{v}, \boldsymbol{v} \rangle | \Delta_1 + \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_1\\ &= \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_1 = \theta_{11}.\end{aligned}$$

This means that (Δ_2, K_2) is a contact manifold such that Ψ_{12} is a contact diffeomorphism. For Δ_i $(i = 3, 4)$, we have the similar calculation, so that (Δ_i, K_i) $(i = 3, 4)$ are contact manifolds such that Ψ_{1i} are contact diffeomorphisms. This completes the proof. $\square$

We can also give contact diffeomorphisms $\Psi_{ij} : \Delta_i \longrightarrow \Delta_j$ for other pairs (i, j) as follows:

$$\begin{aligned}\Psi_{23}(\boldsymbol{v}, \boldsymbol{w}) &= (2\boldsymbol{v} - \boldsymbol{w}, \boldsymbol{v} - \boldsymbol{w}),\\ \Psi_{32}(\boldsymbol{v}, \boldsymbol{w}) &= (\boldsymbol{v} - \boldsymbol{w}, \boldsymbol{v} - 2\boldsymbol{w}),\\ \Psi_{24}(\boldsymbol{v}, \boldsymbol{w}) &= (2\boldsymbol{v} - \boldsymbol{w}, \boldsymbol{w}),\\ \Psi_{42}(\boldsymbol{v}, \boldsymbol{w}) &= \left(\frac{\boldsymbol{v} + \boldsymbol{w}}{2}, \boldsymbol{w} \right),\\ \Psi_{34}(\boldsymbol{v}, \boldsymbol{w}) &= (\boldsymbol{v}, \boldsymbol{v} - 2\boldsymbol{w}),\\ \Psi_{43}(\boldsymbol{v}, \boldsymbol{w}) &= \left(\boldsymbol{v}, \frac{\boldsymbol{v} - \boldsymbol{w}}{2} \right).\end{aligned}$$

We now explain the situation by a "mandala of Legendrian dualities" as the following commutative diagram:



We can also consider the following two extra double fibrations:

(5) (a) $S_r^n \times S_r^n \supset \Delta_5 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\}$,
(b) $\pi_{51} : \Delta_5 \longrightarrow S_r^n$, $\pi_{52} : \Delta_5 \longrightarrow S_r^n$,
(c) $\theta_{51} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_5$, $\theta_{52} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_5$.

(6) (a) $H_{r-1}^n \times H_{r-1}^n \supset \Delta_6 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\}$,
(b) $\pi_{61} : \Delta_6 \longrightarrow H_{r-1}^n$, $\pi_{62} : \Delta_6 \longrightarrow H_{r-1}^n$,
(c) $\theta_{61} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_6$, $\theta_{62} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_6$.

We have the following theorem.

Theorem 3.2. *Under the same notations as the above, each* (Δ_i, K_i) $(i = 5, 6)$ *is a contact manifold and both of* π_{ij} $(j = 1, 2)$ *are Legendrian fibrations.*

The proof of the theorem is almost the same as that for (Δ_1, K_1) in Theorem 3.1. We can show that (Δ_5, K_5) (respectively, (Δ_6, K_6)) is locally diffeomorphic to the projective cotangent bundle $\pi : PT^*S^n_r \longrightarrow S^n_r$ (respectively, $\pi : PT^*H^n_{r-1} \longrightarrow H^n_{r-1}$) which sends K_i to the canonical contact structure. We remark that these contact manifolds (Δ_j, K_j) $(j = 5, 6)$ are not canonically contact diffeomorphic to (Δ_i, K_i) $(i = 1, 2, 3, 4)$. Therefore we cannot add these contact manifolds to the mandala of Legendrian dualities. By definition, S^n_0 is a unit sphere in Euclidean space $\mathbf{R}^{n+1}_0$, so that (Δ_5, K_5) is the well known classical spherical duality in this case. Finally we remark that $\Delta_6 = \emptyset$ in $H^n_0 \times H^n_0$.

4. Applications. In this section we consider differential geometry of hypersurfaces in pseudo-spheres as an application of the Legendrian dualities theorems.

4.1. Pseudo-spheres in $\mathbf{R}^{n+1}_1$. We consider hypersurfaces in pseudo-spheres in Minkowski space. In [7] it has been studied a local extrinsic differential geometry on hypersurfaces in hyperbolic space H^n_0 as an application of Legendrian singularity theory. We give a brief review on the theory. Let $\boldsymbol{X} : U \longrightarrow H^n_0$ be a regular hypersurface (i.e., an embedding), where $U \subset \mathbf{R}^{n-1}$ is an open subset. We denote that $M = \boldsymbol{x}(U)$ and identify M with U through the embedding $\boldsymbol{X}$. Since $\langle \boldsymbol{X}, \boldsymbol{X} \rangle \equiv -1$, we have $\langle \boldsymbol{X}_{u_i}, \boldsymbol{X} \rangle \equiv 0$ $(i = 1, \dots, n-1)$, where $u = (u_1, \dots u_{n-1}) \in U$. This means that $\boldsymbol{X}$ is a timelike unit normal vector field to M in $\mathbf{R}^{n+1}_1$. We can construct a spacelike unit normal $\boldsymbol{e}(u)$ of M in H^n_0 at $p = \boldsymbol{X}(u)$ with the properties $\langle \boldsymbol{e}, \boldsymbol{X}_{u_i} \rangle \equiv \langle \boldsymbol{e}, \boldsymbol{X} \rangle \equiv 0, \langle \boldsymbol{e}, \boldsymbol{e} \rangle \equiv 1$. Therefore the vector $\boldsymbol{X} \pm \boldsymbol{e}$ is lightlike. We define maps $\mathbf{E} : U \longrightarrow S^n_1$ and $\mathbf{L}^\pm : U \longrightarrow LC^*$ by $\mathbf{E}(u) = \boldsymbol{e}(u)$ and $\mathbf{L}^\pm(u) = \boldsymbol{X}(u) \pm \boldsymbol{e}(u)$ which are called the *de Sitter Gauss image* and the *lightcone Gauss image* of M. In order to define curvatures for M, we can use both of the de Sitter Gauss image $\mathbf{E}$ and the lightcone Gauss image $\mathbf{L}^\pm$ like as the Gauss map of a hypersurface in Euclidean space. We can interpret that $d\mathbf{E}(u_0)$ is a linear transformation on T_pM for $p = \boldsymbol{X}(u_0)$. Since the derivative $d\boldsymbol{X}(u_0)$ can be identified with the identity mapping 1_{T_pM} on the tangent space T_pM under the identification of U and M via the embedding $\boldsymbol{X}$, we have

$$d\mathbf{L}^\pm(u_0) = 1_{T_pM} \pm d\mathbf{E}(u_0),$$

so that $d\mathbf{L}^\pm(u_0)$ can be also interpreted as a linear transformation on T_pM. We call the linear transformation $A_p = -d\mathbf{E}(u_0)$ the *de Sitter shape operator* and $S^\pm_p = -d\mathbf{L}^\pm(u_0) : T_pM \longrightarrow T_pM$ the *lightcone shape operator* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{X}(u_0)$. The *de Sitter Gauss-Kronecker curvature* of M at $p = \boldsymbol{X}(u_0)$ is defined to be $K_d(u_0) = \det A_p$ and the *lightcone Gauss-Kronecker curvature* of M at $p = \boldsymbol{X}(u_0)$ is $K^\pm_\ell(u_0) = \det S^\pm_p$.

In [7] we have investigate the geometric meanings of the lightcone Gauss-Kronecker curvature from the contact viewpoint. One of the consequences of the results is that the lightcone Gauss-Kronecker curvature estimates the contact of hypersurfaces with hyperhorospheres. It has been also shown that the Gauss-Bonnet type theorem holds on the (normalized) lightcone Gauss-Kronecker curvature [9]. We emphasize that we discovered a new geometry in hyperbolic space through these research [3,7,9,13] which is called "Horospherical Geometry". We can interpret the above construction by using the Legendrian duality theorem. For any regular hypersurface $\boldsymbol{X} : U \longrightarrow H^n(-1)$, we have $\langle \boldsymbol{X}(u), \mathbf{E}(u) \rangle = 0$. Therefore, we can define a pair of embeddings $\mathcal{L}_1 : U \longrightarrow \Delta_1$ by $\mathcal{L}_1(u) = (\boldsymbol{X}(u), \mathbf{E}(u))$. By definition, $\mathcal{L}_1$ is a Legendrian embedding if and only if $\mathbf{E}$ is a spacelike unit normal vector field along M. Therefore we have the wave front $\mathbf{E}(U) = \pi_{12} \circ \mathcal{L}_1(U)$ of $\mathcal{L}_1(U)$ through the Legendrian fibration π_{12}. On the other hand, by the mandala of Legendrian dualities, $\mathcal{L}_2 = \Psi_{12} \circ \mathcal{L}_1$ is a Legendrian embedding into (Δ_2, K_2), so that we have $\mathcal{L}_2(u) = (\boldsymbol{X}(u), \mathbf{L}^+(u))$. If we consider another normal direction $\mathbf{E}(u) = -\boldsymbol{e}(u)$, then we have $\mathcal{L}_2(u) = (\boldsymbol{X}(u), \mathbf{L}^-(u))$. Moreover, the mandala of the Legendrian dualities gives more information. We have $\mathcal{L}_3 = \Psi_{13} \circ \mathcal{L}_1$ and $\mathcal{L}_4 = \Psi_{14} \circ \mathcal{L}_1$ which are Legendrian embeddings into (Δ_3, K_3) and (Δ_4, K_4) respectively. Especially $\mathcal{L}_4$ is useful for the study of spacelike hypersurfaces in the nullcone (lightcone). Since the induced metric on the nullcone is degenerate, we cannot apply ordinary submanifold theory of semi-Riemannian geometry. In [8] the Legendrian embedding $\mathcal{L}_4$ has been used for the construction of the extrinsic differential geometry on spacelike hypersurfaces in the nullcone. In [17] Kasedou constructed the extrinsic differential geometry on spacelike hypersurfaces in S^n_1 analogous to the theory in [7]. For a spacelike embedding $\boldsymbol{X}: U \longrightarrow S^n_1$, he defined a hyperbolic Gauss image $\mathbf{E}: U \longrightarrow H^n_0$ and a lightcone Gauss image $\mathbf{L}^{\pm}: U \longrightarrow \Lambda^n$ exactly the same way as those in [7]. Of course, some geometric properties are different from those o hypersurfaces in H^n_0. We can also interpret his construction by the mandala of Legendrian dualities similar way as the case for hypersurfaces in H^n_0. However, we have some more information by the mandala of Legendrian dualities. We can start any Legendrian embedding $\mathcal{L}_i : U \longrightarrow \Delta_i$ $(i = 1, 2, 3, 4)$. Here we start $i = 1$. Then we write $\mathcal{L}_1(u) = (\boldsymbol{X}^h(u), \boldsymbol{X}^d(u))$, $M^h = \boldsymbol{X}^h(U)$ and $M^d = \boldsymbol{X}^d(U)$. Since $\mathcal{L}_1$ is a Legendrian embedding, $\boldsymbol{X}^h$ and $\boldsymbol{X}^d$ can be considered as unit normal vector fields each other. If $\boldsymbol{X}^h, \boldsymbol{X}^d$ are immersive, these are exactly unit normal vector fields each other in the ordinary sense, so that we can define the principal curvatures on these hypersurfaces each other. Suppose that both of $\boldsymbol{X}^h$ and $\boldsymbol{X}^d$ are immersive, we denote $\kappa^h_d(u)$ (respectively, $\kappa^d_h(u)$ one of the principal curvatures of M^h (respectively, M^d) with the Gauss image $\boldsymbol{X}^d$ (respectively, $\boldsymbol{X}^h$) at $u \in U$. Under the identification of U with M^h (respectively, M^d) through $\boldsymbol{X}^h$ (respectively, $\boldsymbol{X}^d$), $d\boldsymbol{X}^d$ is the inverse mapping of $d\boldsymbol{X}^h$ vice versa. Therefore we have the following proposition.

Proposition 4.1. *Under the above notation, suppose that both of $\boldsymbol{X}^h$ and $\boldsymbol{X}^d$ are immersive. Then we have the relation $\kappa^h_d(u)\kappa^d_h(u) = 1$.*

By the above proposition, we have nice dual relations between special surfaces in H^3_0 and S^3_1. For example the dual surface of a Linear Weingarten surface in H^3_0 is also a Linear Weingarten surface in S^3_1 vice versa. There appeared several results on such surfaces recently [1,6,18]. Moreover, in [14,16] it has been classified the generic singularities of exceptional Linear Weingarten surfaces in H^3_0 and S^3_1 which are corresponding by the Legendrian duality. There are very beautiful dual relations between these singularities.

4.2. Anti de Sitter space in $\mathbf{R}^{n+1}_2$. In [4,5,15] extrinsic differential geometry on submanifolds in Anti de Sitter space are investigated. In these papers, it has been only considered the case when $n \leq 4$. The detailed arguments for the general dimension case will be appeared in elsewhere. Let $\boldsymbol{X}: U \longrightarrow H^n_1$ be an embedding from an open subset $U \subset \mathbf{R}^{n-1}$. We denote that $M = \boldsymbol{X}(U)$ and identify U and M through $\boldsymbol{X}$. If M is a spacelike hypersurface, we have the timelike unit normal $\boldsymbol{e}(u) \in H^n_1$. In this case, we can apply the Legednrian duality (Δ_6, K_6) and obtain the similar results as those of the classical spherical geometry (cf., [4]). For a timelike hypersurface M, we have the spacelike unit normal $\boldsymbol{e}(u) \in S^n_2$ and the null normal $\mathbf{N}^{\pm}(u) = \boldsymbol{X}(u) \pm \boldsymbol{e}(u) \in \Lambda^n$ like as in [7]. Therefore we have Legedrian embeddings $\mathcal{L}_1 : U \longrightarrow \Delta_1$ and $\mathcal{L}^{\pm}_2 : U \longrightarrow \Delta_2$ defined by $\mathcal{L}_1(u) = (\boldsymbol{X}(u), \boldsymbol{e}(u))$ and $\mathcal{L}^{\pm}_2(u) = (\boldsymbol{X}(u), \mathbf{N}^{\pm}(u))$. The Legendrian embedding $\mathcal{L}_2$ is the most interesting one. Analogous to the previous subsection, we can define the principal curvatures by using the Legendrian duality and study geometric properties of these curvatures as an application of the theory of Legendrian singularities (cf., [5] for $n = 3$). We denote the product of the principal curvatures $K_{AdS^n}(u)$ corresponding to the Legendrian embedding $\mathcal{L}_2$ and call it the *AdS-null Gauss-Kronecker curvature* of M. In this case, we have the following construction: We define the following set

$$S^1_t \times S^{n-2}_s = \{\boldsymbol{x} = (x_1, \cdots, x_{n+1}) \in \Lambda^n \mid x_1^2 + x_2^2 = 1\}.$$

For any $\boldsymbol{x} = (x_1, \cdots, x_{n+1}) \in \Lambda^n$, we have

$$\overline{\boldsymbol{x}} = \frac{1}{\sqrt{x_1^2 + x_2^2}}\,\boldsymbol{x} \in S^1_t \times S^{n-2}_s.$$

We may consider that $S^1_t \times S^{n-2}_s$ is the ideal boundary (i.e., the end) of H^n_1 which has a conformally flat Lorentzian structure. We can define the *Anti de Sitter $S^1_t \times S^{n-2}_s$-Gauss map*

$$\overline{\mathbf{NG}}^{\pm} : U \longrightarrow S^1_t \times S^{n-2}_s$$

by $\overline{\mathbf{NG}}^{\pm}(u) = \overline{\mathbf{N}^{\pm}(u)}$. We define the corresponding Gauss-Kronecker curvature by differentiating the Anti de Sitter $S^1_t \times S^{n-2}_s$-Gauss map. We denote it $\overline{K}_{AdS^n}(u)$ and call the *normalized AdS-null Gauss-Kronecker curvature* of M. We can show that the Anti de Sitter $S^1_t \times S^{n-2}_s$-Gauss map is a Lagrangian map such that the Legendrian embedding $\mathcal{L}_2(U)$ is a covering over the Lagrangian submanifold which is the lift of the Anti de Sitter $S^1_t \times S^{n-2}_s$-Gauss map. Since singularities of $\mathbf{N}^{\pm}$ (respectively, $\overline{\mathbf{NG}}^{\pm}$) are the zero points set of K_{AdS^n} (respectively, $\overline{K}_{AdS^n}$), the singular set of $\mathbf{N}^{\pm}$ and $\overline{\mathbf{NG}}^{\pm}$ are the same. Moreover, in [5] we have shown that generic singularities of $\mathbf{N}^{\pm}$ are the only the cuspidaledge or the swallowtail. The *cuspidaledge* is parame-

cuspidaledge. **swallowtail.**

Fig. 1.

trized by $C = \{(x_1, x_2, x_3) \mid {x_1}^2 = {x_2}^3\}$ and the *swallowtail* is $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ (cf., Fig. 1). By a general result of the theory of Legendrian/Lagrangian singularities, the cuspidaledge (respectively, the swallowtail) of $\mathbf{N}^{\pm}$ is the fold point (respectively, the cusp point) of $\overline{\mathbf{NG}}^{\pm}$ (cf., [2], Part III).

Finally we remark that there is a conjecture in Physics that the classical gravitation theory on AdS^n is equivalent to the conformal field theory on the ideal boundary of AdS^n proposed by Maldacena [19]. It is called the *AdS/CFT-correspondence* or the *holographic principle* [21]. If the conjecture is true, extrinsic geometric properties on submanifolds in AdS^n have corresponding Gage theoretic geometric properties in the ideal boundary $S^1_t \times S^{n-2}_s$. Here, we might say that the Anti de Sitter $S^1_t \times S^{n-2}_s$-Gauss map is one of the analogous notions belonging to the AdS/CFT-correspondence in Mathematics.

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