Rational Diophantine sextuples with mixed signs

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Abstract: A rational Diophantine m-tuple is a set of m nonzero rationals such that the product of any two of them is one less than a perfect square. Recently Gibbs constructed several examples of rational Diophantine sextuples with positive elements. In this note, we construct examples of rational Diophantine sextuples with mixed signs. Indeed, we show that such examples exist for all possible combinations of signs.

Key words: Diophantine sextuples; elliptic curve.

1. Introduction. A set of m nonzero rationals $\{a_1, a_2, \dots, a_m\}$ is called *a* (*rational*) *Diophantine m-tuple* if $a_i a_j + 1$ is a perfect square for all $1 \le i < j \le m$ (see [5]).

The principal question is how large a rational Diophantine tuple can be. In case of integer Diophantine tuples, the corresponding question is almost completely answered. Namely, it is well-known and easy to prove that there exist infinitely many integer Diophantine quadruples (e.g. $\{k-1, k+1, 4k, 16k^3 - 4k\}$ for $k \ge 2$), while it was proved in [8] that there does not exist an integer Diophantine sextuple and there are only finitely many such quintuples (see also [10]). However, in the case of rational Diophantine tuples, no absolute upper bound for the size of such sets is known (the existence of such a bound follows from the Lang conjecture on varieties of general type). The first example of a rational Diophantine quadruple was the set $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ found by Diophantus (see [4]). Euler found infinitely many rational Diophantine quintuples (see [13]). E.g.

$$\left\{1, 3, 8, 120, \frac{777480}{8288641}\right\}.$$

Since 1999, several examples of rational Diophantine sextuples were found by Gibbs [11,12]. The first example was

$$\left\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\right\}.$$

No example of a rational Diophantine septuple is known.

If $\{a_1, a_2, a_3, a_4, a_5\}$ is a rational Diophantine quintuple, we may consider the hyperelliptic curve

$$y^2 = (a_1x + n)(a_2x + n)(a_3x + n)(a_4x + n)(a_5x + n)$$

of genus $g = 2$. Caporaso, Harris and Mazur [2] proved that the Lang conjecture on varieties of general type implies that for $g \ge 2$ the number $B(g, \mathbf{K}) = \max_C |C(\mathbf{K})|$ is finite. Here C runs over all curves of genus g over a number field $\mathbf{K}$, and $C(\mathbf{K})$ denotes the set of all $\mathbf{K}$-rational points on C. Therefore, the number of elements in a rational Diophantine tuple should be bounded by $5 + B(2, \mathbf{Q})$ (and also by $4 + B(4, \mathbf{Q})$, see [14]).

It can be noted that all Gibbs' examples of sextuples contain six positive rationals. Thus, it makes sense to ask if there exist such sextuples with mixed signs. Since $\{a_1, \dots, a_6\}$ is a Diophantine sextuple if and only if $\{-a_1, \dots, -a_6\}$ has the same property, it suffices to find sextuples with exactly one, two and three negative elements.

2. The constructions. In the constructions of rational Diophantine sextuples, we use several techniques. Most of them can be explained in terms of elliptic curves.

If $\{a, b\}$ is a rational Diophantine pair, then $\{a, b, a + b \pm 2\sqrt{ab+1}\}$ is a rational Diophantine triple. Such triples are called *regular*.

Let $\{a, b, c\}$ be a (rational) Diophantine triple. In order to extend this triple to a quadruple, we have to solve the system

(1) $\quad ax + 1 = \square, \quad bx + 1 = \square, \quad cx + 1 = \square.$

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It is a natural idea to assign to the system (1) the elliptic curve

(2) $\mathcal{E}$: $y^2 = (ax+1)(bx+1)(cx+1)$.

There are three rational points on $\mathcal{E}$ of order 2, and also other obvious rational points

$$P = [0,1], \quad S = \left[\frac{1}{abc}, \frac{\sqrt{(ab+1)(ac+1)(bc+1)}}{abc}\right].$$

The x-coordinate of a point $T \in \mathcal{E}(\mathbf{Q})$ satisfies (1) if and only if $T - P \in 2\mathcal{E}(\mathbf{Q})$ (see [6]). It can be verified that $S \in 2\mathcal{E}(\mathbf{Q})$. This implies that the numbers $x(P \pm S)$ satisfy the system (1). These numbers are exactly the numbers

$$d_{+,-} = a+b+c+2abc \pm 2\sqrt{(ab+1)(ac+1)(bc+1)}$$

obtained by Arkin, Hoggatt and Strauss [1]. They proved that $\{a,b,c,d_+\}$ and $\{a,b,c,d_-\}$ are rational Diophantine quadruples (if their elements are distinct and nonzero). Quadruples of this form are called *regular*. The conjecture is that all integer Diophantine quadruples are regular. Note that if $\{a,b,c\}$ is a regular triple, then $d_+d_- = 0$.

Let $\{a,b,c,d\}$ be a rational Diophantine quadruple such that $abcd \neq 1$ and let

$$e_{+,-} = \frac{(a+b+c+d)(abcd+1)}{(abcd-1)^2} + \frac{2abc+2abd+2acd+2bcd}{(abcd-1)^2} \pm \frac{2\sqrt{(ab+1)(ac+1)(ad+1)(bc+1)(bd+1)(cd+1)}}{(abcd-1)^2}.$$

In [5] we proved that $\{a,b,c,d,e_+\}$ and $\{a,b,c,d,e_-\}$ are rational Diophantine quintuples (if their elements are distinct and nonzero). Note that if $\{a,b,c,d\}$ is a regular quadruple, then $e_+e_- = 0$ (see [5, Proposition 2]). A rational Diophantine quintuple $\{a,b,c,d,e\}$ is called *regular* if it is obtained by the construction from [5] or, equivalently, if

$$(abcde+2abc+a+b+c-d-e)^2 = 4(ab+1)(ac+1)(bc+1)(de+1).$$

This construction can be explained also in the terms of elliptic curve $\mathcal{E}$. Namely, let D be the point on $\mathcal{E}$ with the x-coordinate d. Then the numbers e_+ and e_- are exactly the x-coordinates of the points $D \pm S$ on $\mathcal{E}$. (See [7] for the characterization of regular quadruples and quintuples in terms of the elliptic curve $y^2 = (ax+1)(bx+1)(cx+1)(dx+1)$.)

Now we describe briefly three techniques for contruction of rational Diophantine sextuples.

- Let $ab+1 = r^2$ and $c = a+b+2r$, i.e. take $\{a,b,c\}$ to be a regular rational Diophantine triple. Consider the elliptic curve $\mathcal{E}$ given by (2). We may expect that it has infinitely many rational points (see [9]). Of course, we can test only finitely many such points. The test will involve the condition that certain rational number, with randomly-looking numerator, is a perfect square, which is more likely to be satisfied if the numerator is small. Thus, in this construction we use rational points of relatively small heights on $\mathcal{E}$. For example, if $\mathrm{rank}(\mathcal{E}(\mathbf{Q})) = 2$ and X_1, X_2 are generators of $\mathcal{E}(\mathbf{Q})/\mathcal{E}(\mathbf{Q})_{\mathrm{tors}}$, we consider the points of the form $T = m_1X_1 + m_2X_2$, for $|m_i| \in \{0,1,2,3\}$. If $D = P + 2T = [d,d']$, then $\{a,b,c,d\}$ is a rational Diophantine quadruple (at least some of these quadruples are irregular; this is why we prefer to avoid curves with rank 1). Define the points $E = D + S = [e,e']$ and $F = D - S = [f,f']$. Then $\{a,b,c,d,e\}$ and $\{a,b,c,d,f\}$ are rational Diophantine quintuples (and if $\{a,b,c,d\}$ was irregular, then $ef \neq 0$). If $ef+1$ is a perfect square, then $\{a,b,c,d,e,f\}$ is a rational Diophantine sextuple (assuming that all its elements are distinct and nonzero). In that way, we find e.g. the sextuple $\{\frac{5}{36}, \frac{5}{4}, \frac{32}{9}, \frac{189}{4}, \frac{665}{1521}, \frac{3213}{676}\}$ with positive elements (found already by Gibbs [12]), but also several sextuples with mixed signs, e.g. $\{\frac{5}{14}, \frac{7}{2}, \frac{48}{7}, \frac{1680}{361}, -\frac{2310}{19321}, \frac{93840}{71407}\}$.
- Take again the regular triple $\{a,b,c\}$, where $c = a+b+2r$, and apply the same construction to obtain a regular triple $\{b,c,d\}$. We find that $d = a+4b+4r$. The only remaining condition in order that $\{a,b,c,d\}$ be a Diophantine quadruple is that $ad+1$ is a perfect square. This condition leads to $(a+2r)^2 - 3 = \square$, and it is satisfied if we take $r = \frac{u-a}{2}$, where $u = \frac{\alpha^2+3}{2\alpha}$ for $\alpha \in \mathbf{Q}$.
  Applying the construction from [5] to the quadruple $\{a,b,c,d\}$, we obtain the quintuples $\{a,b,c,d,e_+\}$ and $\{a,b,c,d,e_-\}$. If e_+e_-+1 is a perfect square, then $\{a,b,c,d,e_+,e_-\}$ is a rational Diophantine sextuple (assuming that all its elements are distinct and nonzero). As an example of a sextuple obtained by this construction, we give $\{\frac{27}{35}, -\frac{35}{36}, \frac{1007}{1260}, -\frac{352}{315},$

$\frac{72765}{106276}, -\frac{5600}{4489}\}$.

- Let $\{a, b\}$ be a rational Diophantine pair. For a rational number t, define $c = -\frac{4t(-1+t)(bt-a)}{(-a+bt^2)^2}$. It is easy to check that $ac+1$ and $bc+1$ are perfect squares, and therefore $\{a, b, c\}$ is a rational Diophantine triple. We can extend this triple to an (irregular) quadruple by $d = \frac{8(c-a-b)(a+c-b)(b+c-a)}{(a^2+b^2+c^2-2ab-2ac-2bc)^2}$ (see [5, Proposition 3]). This number is the x-coordinate of the point $3P$ on $\mathcal{E}$. Again, we can apply the construction from [5] to the quadruple $\{a, b, c, d\}$, to obtain e_+ and e_-, and if $e_+e_- + 1$ is a perfect square, then we get a rational Diophantine sextuple $\{a, b, c, d, e_+, e_-\}$ (provided that all its elements are distinct and nonzero). The reason why in this construction we use irregular triples $\{a, b, c\}$ is that for regular triples, we have $d = d_+$, so the resulted quadruple is regular and gives $e_+e_- = 0$. By this construction we find e.g. the sextuple $\{-\frac{5}{9}, \frac{32}{45}, \frac{27}{20}, \frac{216032}{937445}, -\frac{185232905}{263802564}, \frac{175578975}{136095556}\}$.

The described algorithms are implemented in PARI/GP [15], and for computing the ranks we use MWRANK [3].

3. Examples. We give the list of 26 rational Diophantine sextuples with mixed signs obtained by the constructions described in the previous section.

By the first method we obtain the following rational Diophantine sextuples:

$$\left\{\frac{19}{12}, \frac{33}{4}, \frac{52}{3}, \frac{60}{2209}, -\frac{495}{24964}, \frac{595}{12}\right\},$$
$$\left\{\frac{31}{84}, \frac{9}{7}, \frac{49}{12}, \frac{160}{21}, -\frac{455}{3468}, \frac{7200}{2023}\right\},$$
$$\left\{\frac{5}{14}, \frac{7}{2}, \frac{48}{7}, \frac{1680}{361}, -\frac{2310}{19321}, \frac{93840}{71407}\right\},$$
$$\left\{\frac{7}{40}, -\frac{75}{56}, \frac{41}{70}, -\frac{5376}{4805}, -\frac{300288}{241115}, \frac{165}{224}\right\},$$
$$\left\{\frac{5}{24}, -\frac{64}{15}, -\frac{407}{120}, -\frac{1530}{361}, \frac{2088}{9245}, \frac{399245}{2889816}\right\},$$
$$\left\{-\frac{8}{17}, \frac{85}{72}, -\frac{763}{1224}, \frac{18360}{11449}, \frac{4914}{8993}, \frac{332605}{496008}\right\},$$
$$\left\{-\frac{5}{33}, \frac{121}{60}, \frac{131}{660}, \frac{171360}{30899}, \frac{51978528}{54014455}, \frac{8041}{1500}\right\},$$
$$\left\{\frac{8}{23}, \frac{161}{72}, \frac{8695}{1656}, \frac{54648}{22201}, -\frac{11270}{62001}, \frac{46288935}{9481336}\right\}.$$

We note that the sextuple $\{\frac{31}{84}, \frac{9}{7}, \frac{49}{12}, \frac{160}{21}, -\frac{455}{3468}, \frac{7200}{2023}\}$ is rediscovered by the second method. By the second method we also find the following rational Diophantine sextuples:

$$\left\{\frac{147}{20}, \frac{25}{28}, \frac{96}{35}, -\frac{11}{140}, \frac{30723}{3380}, \frac{165}{1183}\right\},$$
$$\left\{\frac{253}{1140}, -\frac{9}{380}, \frac{125}{57}, \frac{247}{60}, \frac{6688}{375}, \frac{2016}{95}\right\},$$
$$\left\{\frac{27}{35}, -\frac{35}{36}, \frac{1007}{1260}, -\frac{352}{315}, \frac{72765}{106276}, -\frac{5600}{4489}\right\},$$

Finally, we list the rational Diophantine sextuples found by the third method:

$$\left\{\frac{1}{6}, \frac{27}{8}, \frac{385}{96}, \frac{1280}{243}, \frac{250705}{44376}, -\frac{25415}{161376}\right\},$$
$$\left\{\frac{27}{14}, \frac{49}{18}, -\frac{16}{63}, \frac{269654}{113569}, \frac{11572496}{19969047}, -\frac{15578784}{44488087}\right\},$$
$$\left\{\frac{24}{35}, -\frac{75}{56}, \frac{77}{120}, -\frac{846600}{634207}, \frac{5629624}{7540215}, -\frac{4456963}{3346680}\right\},$$
$$\left\{\frac{5}{9}, -\frac{27}{20}, -\frac{55}{36}, \frac{96305}{158404}, \frac{23144992}{59202405}, -\frac{31157568}{20220605}\right\},$$
$$\left\{\frac{5}{9}, -\frac{27}{20}, \frac{13}{20}, -\frac{304083}{212180}, \frac{20055200}{31573161}, -\frac{79520320}{67125249}\right\},$$
$$\left\{-\frac{5}{9}, \frac{27}{20}, \frac{32}{45}, \frac{216032}{937445}, -\frac{185232905}{263802564}, \frac{175578975}{136095556}\right\},$$
$$\left\{\frac{27}{11}, \frac{77}{36}, -\frac{32}{99}, -\frac{43424}{2297339}, \frac{811864053}{368716804}, -\frac{808311427}{2102956164}\right\},$$
$$\left\{\frac{21}{22}, \frac{33}{56}, -\frac{64}{77}, -\frac{3340352}{3625853}, \frac{1092049959}{1018087688}, -\frac{778578801}{1587999368}\right\},$$
$$\left\{\frac{27}{35}, -\frac{35}{36}, \frac{161}{180}, -\frac{4771879}{4287380}, \frac{917801280}{4823805007}, -\frac{2117588000}{6213359943}\right\},$$
$$\left\{\frac{5}{28}, -\frac{27}{35}, \frac{35}{36}, \frac{3838005}{64606108}, \frac{324705510976}{300303876645}, -\frac{329539009184}{358699363245}\right\},$$
$$\left\{\frac{7}{26}, -\frac{221}{72}, -\frac{297}{104}, \frac{226791}{1867424}, \frac{18453763328}{60529284729}, -\frac{19040799232}{6576074649}\right\},$$
$$\left\{-\frac{14}{45}, \frac{135}{56}, -\frac{185}{504}, \frac{25432135}{14622776}, \frac{11585718144}{50291423405}, \frac{314271141184}{117352732005}\right\},$$
$$\left\{\frac{14}{45}, -\frac{135}{56}, -\frac{832}{315}, \frac{21739328}{125951315}, \frac{197932494375}{570623898632}, -\frac{207609892105}{76457704968}\right\},$$
$$\left\{-\frac{14}{45}, \frac{77}{40}, \frac{135}{56}, \frac{203687253}{361681960}, -\frac{5323853454400}{12959750399967}, \frac{4826209930880}{3371383988343}\right\},$$
$$\left\{-\frac{7}{17}, -\frac{425}{1008}, \frac{2432}{1071}, \frac{80888528768}{50503742919}, \frac{1661966668042065}{1421147949949456}, \frac{13748985346416705}{5799449383741456}\right\}.$$

4. Curves with the rank 8. The examples of rational Diophantine sextuples found by Gibbs were used in [7] and [9] to find examples of elliptic curves of the form

$$y^2 = (ax+1)(bx+1)(cx+1)(dx+1), \tag{3}$$

where $\{a, b, c, d\}$ is a Diophantine quadruple, and

$$y^2 = (ax+1)(bx+1)(cx+1), \tag{4}$$

where $\{a, b, c\}$ is a Diophantine triple, with relatively large rank. In both cases, examples with rank equal to 8 were found. Using the examples from the previous section, i.e. taking $\{a, b, c, d\}$ and $\{a, b, c\}$ to be subquadruples and subtriples of Diophantine sextuples, we can find more examples with the same property. Indeed, we have found by MWRANK that the curve (3) has rank 8 for

$$\{a, b, c, d\} = \left\{\frac{385}{96}, \frac{1280}{243}, \frac{250705}{44376}, -\frac{25415}{161376}\right\},$$

while the curve (4) has rank 8 for $\{a, b, c\}$ equal to

$$\left\{-\frac{1530}{361}, \frac{2088}{9245}, \frac{399245}{2889816}\right\},$$

$$\left\{\frac{8695}{1656}, \frac{54648}{22201}, \frac{46288935}{9481336}\right\},$$

$$\left\{\frac{8695}{1656}, -\frac{11270}{62001}, \frac{46288935}{9481336}\right\},$$

$$\left\{\frac{21}{22}, \frac{1092049959}{1018087688}, -\frac{778578801}{1587999368}\right\},$$

$$\left\{\frac{96305}{158404}, \frac{23144992}{59202405}, -\frac{31157568}{20220605}\right\},$$

$$\left\{\frac{269654}{113569}, \frac{11572496}{19969047}, -\frac{15578784}{44488087}\right\}.$$

The ranks have been computed unconditionally, except for the last two triples where MWRANK gives that the rank is equal to 7 or 8, while the Parity Conjeture gives that the rank is even.

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