Explicit quasiconformal extensions and Löwner chains

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Abstract: In this paper we construct Löwner chains which enable us to derive quasiconformal extension criteria for typical classes of univalnet functions. This method also provides us explicit quasiconformal extensions.

Key words: Univalent functions; quasiconformal mapping; Löwner(Loewner) chains.

1. Introduction. Let $\mathcal{A}$ be the class of analytic functions f on the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ normalized so that $f(0) = f'(0) - 1 = 0$ and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions univalent in $\mathbf{D}$. We say that $f \in \mathcal{S}(k)$, $0 \le k < 1$, if $f \in \mathcal{S}$ and f has a quasiconformal extension on the complex plane $\mathbf{C}$ so that the complex dilatation $\mu_f = f_{\bar z}/f_z$ satisfies $|\mu_f(z)| \le k$ almost every z in $\mathbf{C}$, where $f_z = \partial f/\partial z$, $f_{\bar z} = \partial f/\partial \bar z$. Let $U(k)$ be the closed hyperbolic disk in the right half plane centered at 1 with radius arctanh k, $0 \le k < 1$, i.e.

$$U(k) = \left\{ w \in \mathbf{C} : \left| \frac{w-1}{w+1} \right| \le k \right\} = \left\{ w \in \mathbf{C} : \left| w - \frac{1+k^2}{1-k^2} \right| \le \frac{2k}{1-k^2} \right\}.$$

Note that $w \in U(k)$ if and only if $1/w \in U(k)$.

The following is known;

Theorem A. *For $f \in \mathcal{A}$, let $h(z)$ represent one of the quantities $zf'(z)/f(z)$, $1 + zf''(z)/f'(z)$ and $f'(z)$. If*

$$h(z) \in U(k) \tag{1}$$

for all $z \in \mathbf{D}$, then the function f can be extended to a k-quasiconformal automorphism of $\mathbf{C}$.

Brown [3] established this theorem for the case when $h(z) = zf'(z)/f(z)$ by giving an explicit quasiconformal extension and Padmanabhan and Kumar [9] for the case when $h(z) = f'(z)$ with a Löwner chain. Sugawa [12] presented a general method producing the above quasiconformal extension criteria (1) by constructing a suitable holomorphic motion and applying the optimal λ-lemma. Sugawa's method, however, does not give explicit quasiconformal extensions.

On the other hand, Becker [1] provided quasiconformal extensions of univalent functions by Löwner chains. In this paper, we construct Löwner chains to prove Theorem A by employing Becker's theorem. Some more results are also obtained in Section 4, which include quasiconformal extensibility for subclasses of close-to-convex functions, Bazilevič functions and α-convex functions.

2. Preliminaries. A function $f(z,t) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n$ defined on $\mathbf{D} \times [0,\infty)$ is called a *Löwner chain* if $f_t(z) = f(z,t)$ is holomorphic and univalent in $\mathbf{D}$ for each $t \in [0,\infty)$ and satisfies $f_s(\mathbf{D}) \subsetneq f_t(\mathbf{D})$ for $0 \le s < t < \infty$.

The following necessary and sufficient condition for a Löwner chain is well known [10];

Theorem B ([10], Theorem 6.2)**.** *The function $f(z,t)$ is a Löwner chain if and only if the following two conditions are satisfied;*

1. *The function $f(z,t)$ is analytic in $\mathbf{D}_{r_0} = \{|z| < r_0\}$ for each $t \in [0,\infty)$, absolutely continuous in $t \in [0,\infty)$ for each $z \in \mathbf{D}_{r_0}$ and satisfies*

$$|f(z,t)| \le K_0 e^t \quad (z \in \mathbf{D}_{r_0},\ t \in [0,\infty)) \tag{2}$$

for some positive constants K_0 and $r_0 \in (0,1)$.

2. *There exists a function $p(z,t)$ analytic in $\mathbf{D}$ and measurable in $t \in [0,\infty)$ satisfying*

$$\operatorname{Re} p(z,t) > 0 \quad (z \in \mathbf{D},\ t \in [0,\infty))$$

such that

$$\dot f(z,t) = zf'(z,t)p(z,t) \quad (z \in \mathbf{D}_{r_0},\ t \in [0,\infty)) \tag{3}$$

for almost all t, where $\dot f = \partial f/\partial t$ and $f' = \partial f/\partial t$.

The next theorem due to Becker plays a crucial role in our investigations;

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Theorem C ([1], see also [2]). *Suppose that $f_t(z)$ is a Löwner chain for which $p(z,t)$ in (3) satisfies the condition*

$$p(z,t) \in U(k),\ z \in \mathbf{D}, \text{ a.e.} t \in [0,\infty).$$

Then $f_t(z)$ admits a continuous extension to $\overline{\mathbf{D}}$ for each $t \geq 0$ and the map defined by

$$\hat{f}(re^{i\theta}) = \begin{cases} f(re^{i\theta}, 0), & r < 1, \\ f(e^{i\theta}, \log r), & r \geq 1, \end{cases} \tag{4}$$

is a k-quasiconformal extension of f_0 to $\mathbf{C}$.

3. Löwner chains and extended functions. We shall give the following theorem which implies Theorem A as a corollary;

Theorem 1. *For $f \in \mathcal{A}$, let $h(z)$ represent one of the quantities $zf'(z)/f(z)$, $1 + zf''(z)/f'(z)$ and $f'(z)$. If $h(z) \in U(k)$ for all $z \in \mathbf{D}$, then the function f can be extended to a k-quasiconformal automorphism $\hat{f}$ of $\mathbf{C}$ by setting*

$$\hat{f}(w) = \begin{cases} f(w), & |w| < 1, \\ |w|\, f\left(\dfrac{w}{|w|}\right), & |w| \geq 1, \end{cases} \tag{5}$$

if $h(w) = wf'(w)/f(w)$,

$$\hat{f}(w) = \begin{cases} f(w), & |w| < 1, \\ f\left(\dfrac{w}{|w|}\right) + w\, f'\left(\dfrac{w}{|w|}\right)\left(1 - \dfrac{1}{|w|}\right), & |w| \geq 1, \end{cases} \tag{6}$$

if $h(w) = 1 + wf''(w)/f'(w)$ and

$$\hat{f}(w) = \begin{cases} f(w), & |w| < 1, \\ f\left(\dfrac{w}{|w|}\right) + w\left(1 - \dfrac{1}{|w|}\right), & |w| \geq 1, \end{cases} \tag{7}$$

if $h(w) = f'(w)$.

Proof. Set

$$f_t(z) = e^t f(z). \tag{8}$$

Then we have

$$\frac{1}{p(z,t)} = \frac{zf'(z)}{f(z)}.$$

Since the assumption implies $\mathrm{Re}\{zf'(z)/f(z)\} > 0$, we can see from Theorem 3 that (8) is a Löwner chain. Furthermore, by the assumption and applying Theorem C we conclude the assertion for the case when $h(z) = zf'(z)/f(z)$.

If $h(z) = 1 + zf''(z)/f'(z)$, then we set

$$f_t(z) = f(z) + (e^t - 1)zf'(z). \tag{9}$$

This yields

$$\frac{1}{p(z,t)} = 1 + \left(1 - \frac{1}{e^t}\right)\frac{zf''(z)}{f'(z)} = e^{-t}\cdot 1 + (1 - e^{-t})h(z).$$

Since $0 < e^{-t} \leq 1$ for $t \in [0,\infty)$, we see that the above $1/p(z,t)$ lies in the right half plane, in particular convex set $U(k)$ for all $t \in [0,\infty)$ if $1 + zf''(z)/f'(z) \in U(k)$.

Finally, set

$$f_t(z) = f(z) + z(e^t - 1). \tag{10}$$

Then

$$\frac{1}{p(z,t)} = \frac{1}{e^t} f'(z) + \left(1 - \frac{1}{e^t}\right) = e^{-t}h(z) + (1 - e^{-t})\cdot 1$$

which implies the assertion for the case when $h(z) = f'(z)$. Explicit extensions (5), (6) and (7) are obtained immediately by applying (4) to (8), (9) and (10). □

Remark 3-1. The explicit extension (5) appears in [3] and (7) is in [7]. The Löwner chain (10) is in [9]. On the other hand, (6) and (9) are new.

Actually, we can sometimes detect a Löwner chain from an explicit quasiconformal extension. Let $U'(k) = \{w \in \mathbf{C} : |w - 1| \leq k\}$ for $0 \leq k < 1$. In [4], they have shown that *for $f \in \mathcal{A}$, if $f'(z) \in U'(k)$ then f has a k-quasiconformal extension* by giving the explicit extension

$$\hat{f}(w) = \begin{cases} f(w), & |w| < 1, \\ f\left(\dfrac{1}{\overline{w}}\right) + w - \dfrac{1}{\overline{w}}, & |w| \geq 1. \end{cases}$$

(4) says that an explicit quasiconformal extension $F(w)$ is given by substituting $w/|w|$ for z and $\log|w|$ for t in $f(z,t)$, where $w \in \mathbf{D}^* = \{|w| > 1\}$. The correspondence can be described schematically by

$$\begin{pmatrix} e^t z \\ e^{-t} z \end{pmatrix} \Longleftrightarrow \begin{pmatrix} w \\ 1/\overline{w} \end{pmatrix}. \tag{11}$$

Using these substitutions, we now obtain the Löwner chain

$$f_t(z) = f(e^{-t}z) + z(e^t - e^{-t}),$$

and by Theorem C we have the same quasiconformal extension criterion as in [4]. Similarly, we can see that the explicit function

$$\hat{f}(w) = \begin{cases} f(w), & |w| < 1, \\ |w|^2 f\left(\dfrac{1}{\overline{w}}\right), & |w| \geq 1, \end{cases}$$

generates a Löwner chain

$$f_t(z) = e^{2t} f(e^{-t}z)$$

under the assumption $zf'(z)/f(z) \in U'(k)$, and

$$\hat{f}(w) = \begin{cases} f(w), & |w| < 1, \\ \left(\dfrac{1}{\overline{w}}\right) + f'\left(\dfrac{1}{\overline{w}}\right)\left(w - \dfrac{1}{\overline{w}}\right), & |w| \geq 1, \end{cases}$$

generates

$$f_t(z) = f(e^{-t}z) + zf'(e^{-t}z)(e^t - e^{-t}) \qquad (12)$$

under the assumption $1 + zf''(z)/f'(z) \in U'(k)$ respectively, here the form in (12) can be found in [1].

Remark 3-2. The above three criteria are weaker than Theorem 1.

Remark 3-3. If the substitutions (11) are valid, then a calculation shows equivalence of

$$\left|\frac{(\partial f_t(z)/\partial t) - z(\partial f_t(z)/\partial z)}{(\partial f_t(z)/\partial t) + z(\partial f_t(z)/\partial z)}\right| \leq k \quad \textit{and}$$

$$\left|\frac{\partial F(w)/\partial \overline{w}}{\partial F(w)/\partial w}\right| \leq k.$$

4. Other results. We shall give some more results which relate to the previous section. A sufficient condition for univalence can be deduced from (9);

Theorem 2. *Let $f \in \mathcal{A}$ be a convex function, i.e., f satisfies that* $\mathrm{Re}\,\{1 + zf''(z)/f'(z)\} > 0$ *for all* $z \in \mathbf{D}$. *Then for arbitrary $\alpha \in [0,1]$, the convex combination*

$$\alpha f(z) + (1-\alpha) zf'(z)$$

is univalent in $\mathbf{D}$.

Proof. It is clear for the case $\alpha = 0$. We have seen in the proof of Theorem 1 that (9) is a Löwner chain if $f \in \mathcal{A}$ is a convex function. In particular, $f_t(z)$ is univalent in $\mathbf{D}$ for each $t \in [0, \infty)$, and so is $e^{-t} f_t(z)$. Therefore we obtain our theorem for the case $0 < \alpha \leq 1$ if we put $\alpha = e^{-t}$. □

Next, we will show the following theorem concerning quasiconformal extensibility for close-to-convex functions;

Theorem 3. *Let $f \in \mathcal{A}$. Assume that there exists a $g \in \mathbf{S}$ such that $zg'(z)/g(z) \in U(k)$ and $zf'(z)/g(z) \in U(k)$ for all $z \in \mathbf{D}$. Then f can be extended to a k-quasiconformal automorphism $\hat{F}$ of* $\mathbf{C}$ *by setting*

$$\hat{F}(w) = \begin{cases} f(w), & |w| < 1, \\ f\left(\dfrac{w}{|w|}\right) + (|w| - 1) g\left(\dfrac{w}{|w|}\right), & |w| \geq 1. \end{cases}$$

Proof. Let

$$f_t(z) = f(z) + (e^t - 1) g(z). \qquad (13)$$

Then we have

$$\frac{1}{p(z,t)} = \frac{1}{e^t}\frac{zf'(z)}{g(z)} + \left(1 - \frac{1}{e^t}\right)\frac{zg'(z)}{g(z)}.$$

We can see that $1/p(z,t) \in U(k)$ from the assumption. Consequently, it follows from Theorem B that (13) is a Löwner chain and from Theorem C that f has a k-quasiconformal extension. □

Remark 4-1. The chain (13) appears in [10, p. 52].

Remark 4-2. A similar result can be found in [6] though they do not mention the dilatation k.

Remark 4-3. Theorem 3 includes Theorem 1. Indeed, $g(z) = z, g(z) = f(z)$ and $g(z) = zf'(z)$ corresponds to the case $h(z) = f'(z), h(z) = zf'(z)/f(z)$ and $h(z) = 1 + zf''(z)/f'(z)$, respectively.

Finally, another condition for quasiconformal extensibility is given by;

Theorem 4. *Let $\alpha > 0$. For $f \in \mathcal{A}$, if*

$$\frac{1}{\alpha}\left[1 + \frac{zf''(z)}{f'(z)} + (\alpha - 1)\frac{zf'(z)}{f(z)}\right] \in U(k)$$

for all $z \in \mathbf{D}$, then f can be extended to a k-quasiconformal automorphism of $\mathbf{C}$.

Proof. Set

$$f(z,t) = f(z)\left(1 + (e^{\alpha t} - 1)\frac{zf'(z)}{f(z)}\right)^{1/\alpha}. \qquad (14)$$

Then the calculation shows

$$\frac{1}{p(z,t)} = \frac{1}{e^{\alpha t}} + \frac{1}{\alpha}\left(1 - \frac{1}{e^{\alpha t}}\right) \left(1 + \frac{zf''(z)}{f'(z)} + (\alpha - 1)\frac{zf'(z)}{f(z)}\right)$$

and therefore the theorem follows from the assumption, Theorem B and Theorem C. □

Remark 4-4. A chain similar to (14) appears in [11].

Remark 4-5. Theorem 4 claims the quasiconformal extensibility for a subclass of Bazilevič functions of type α which includes $1/\alpha$-convex functions. Here, we say that $f \in \mathcal{A}$ is an *α-convex function* (e.g. [8, p. 10]) if

$$\mathrm{Re}\left\{\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)\right\} > 0 \ (z \in \mathbf{D}),$$

and $f \in \mathcal{A}$ is a *Bazilevič function of type α* if

$$f(z) = \left[\alpha \int_0^z h(\zeta)g(\zeta)^\alpha \zeta^{-1} d\zeta\right]^{1/\alpha}$$

for a starlike function $g \in \mathcal{A}$ and analytic function h with $h(0) = 1$ satisfying $Re(e^{i\lambda}h) > 0$ in $\mathbf{D}$ for some $\lambda \in \mathbf{R}$. It is known that

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)} + (\alpha - 1)\frac{zf'(z)}{f(z)}\right\} > 0 \ \ (z \in \mathbf{D}) \quad (15)$$

is sufficient for $f \in \mathcal{A}$ to be a Bazilevič function of type α (e.g. [5]). It is easy to see that f is $1/\alpha$-convex and Bazilevič of type α under the assumption of Theorem 4.

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