A new class of nonassociative algebras with involution

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Abstract: This article is devoted to introduce a new class of nonassociative algebras with involution including the class of structurable algebras.

Key words: Nonassociative algebras; Lie (super)algebras.

1. Introduction. Our start point briefly described in a historical setting is the construction of Lie (super)algebras starting from a class of nonassociative algebras. Hence within the general framework of (ϵ, δ)-Freudenthal Kantor triple systems ($\epsilon, \delta = \pm 1$) and the standard embedding Lie (super)algebra construction studied in [5,6,9–12] (see also references therein) we define δ-structurable algebras as a class of nonassociative algebras with involution which coincides with the class of structurable algebras for $\delta = 1$ as introduced and studied in [1,2]. Structurable algebras are a class of nonassociative algebras with involution that include Jordan algebras (with trivial involution), associative algebras with involution, and alternative algebras with involution. They are related to generalized Jordan triple systems of second order (or $(-1, 1)$-Freudenthal Kantor triple systems) as introduced and studied in [14,15] and further studied in [3,4,13,18–21] (see also references therein). Their importance lies with constructions of five graded Lie algebras

$$L(\epsilon, \delta) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2, \quad [L_i, L_j] \subseteq L_{i+j}.$$

For $\delta = -1$ the anti-structurable algebras defined here are a new class of nonassociative algebras that may similarly shed light on the notion of $(-1, -1)$-Freudenthal Kantor triple systems hence (by [5,6]) on the construction of Lie superalgebras and Jordan algebras as it will be shown.

Throughout the paper it is assumed that $(\mathcal{A}, ^-)$ is a finite dimensional nonassociative unital algebra with involution (involutive anti-automorphism, i.e. $\overline{\overline{x}} = x$ and $\overline{xy} = \overline{y}\,\overline{x}$ for $x, y \in \mathcal{A}$) over a field $\mathbf{F}$, $\mathrm{char}\mathbf{F} \neq 2$ or 3. The identity element of $\mathcal{A}$ is denoted by 1. Since $\mathrm{char}\mathbf{F} \neq 2$, by [1] we have $\mathcal{A} = \mathcal{H} \oplus \mathcal{S}$, where $\mathcal{H} = \{a \in \mathcal{A} | \overline{a} = a\}$ and $\mathcal{S} = \{a \in \mathcal{A} | \overline{a} = -a\}$.

Suppose $x, y, z \in \mathcal{A}$. Put $[x, y] := xy - yx$ and $[x, y, z] := (xy)z - x(yz)$. Note that

$$\overline{[x, y, z]} = -[\overline{z}, \overline{y}, \overline{x}]. \tag{1}$$

The operators L_x and R_x are defined by

$$L_x(y) := xy, R_x(y) := yx.$$

2. $\boldsymbol{\delta}$-structurable algebras. For $\delta = \pm 1$ and $x, y \in \mathcal{A}$ define

$$^{\delta}V_{x,y} := L_{L_x(\overline{y})} + \delta(R_x R_{\overline{y}} - R_y R_{\overline{x}}), \tag{2}$$

$$^{\delta}B_{\mathcal{A}}(x, y, z) := {}^{\delta}V_{x,y}(z) = (x\overline{y})z + \delta[(z\overline{y})x - (z\overline{x})y], x, y, z \in \mathcal{A}. \tag{3}$$

$^{+}B_{\mathcal{A}}(x, y, z)$ is called the *triple system obtained from the algebra* $(\mathcal{A}, ^-)$. We will call $^{-}B_{\mathcal{A}}(x, y, z)$ the *anti-triple system obtained from the algebra* $(\mathcal{A}, ^-)$. We shall write for short

$$V_{x,y} := {}^{\delta}V_{x,y}, \quad B_{\mathcal{A}} := ({}^{\delta}B_{\mathcal{A}}, \mathcal{A}).$$

Remark. The upper left index notation is chosen in order not to be mixed with the upper right index notation of [1] which has a different meaning.

A unital non-associative algebra with involution $(\mathcal{A}, ^-)$ is called a *structurable algebra* if the following identity is fulfilled

$$[V_{u,v}, V_{x,y}] = V_{V_{u,v}(x),y} - V_{x,V_{v,u}(y)}, \tag{4}$$

for $V_{u,v} = {}^{+}V_{u,v}, V_{x,y} = {}^{+}V_{x,y}, u, v, x, y \in \mathcal{A}$, and we will call $(\mathcal{A}, ^-)$ an *anti-structurable algebra* if the identity (4) is fulfilled for $V_{u,v} = {}^{-}V_{u,v}, V_{x,y} = {}^{-}V_{x,y}$.

If $(\mathcal{A}, ^-)$ is structurable then, in the terminology of [15], the triple system $B_{\mathcal{A}}$ is called a *generalized Jordan triple system* (abbreviated GJTS) and by [7], $B_{\mathcal{A}}$ is a GJTS *of 2-nd order*, i.e. satisfies the identities (14) and (15). If $(\mathcal{A}, ^-)$ is

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anti-structurable then we call $B_{\mathcal{A}}$ an *anti-GJTS*.

Put $T_x := V_{x,1}$ for $x \in \mathcal{A}$. Then, by (2),

$$T_x = L_x + \delta R_{x-\overline{x}} \tag{5}$$

for $x \in \mathcal{A}$. In particular, $T_h = L_h$ for $h \in \mathcal{H}$.

Remarks. (i) If $u = h \in \mathcal{H}$ and $x, y \in \mathcal{A}$, (4) becomes

$$[L_h, V_{x,y}] = V_{hx,y} - V_{x,hy}. \tag{6}$$

The identity (6) written in element form is

$$\begin{aligned}((hx)\overline{y})z - h((x\overline{y})z) + \delta[((hz)\overline{y})x -\\ h((z\overline{y})x) - ((hz)\overline{x})y + h((z\overline{x})y)] =\\ (x(\overline{y}h))z - (x\overline{y})(hz) + \delta[(z(\overline{y}h))x -\\ (z\overline{y})(hx) + (z(\overline{x}h))y - (z\overline{x})(hy)],\end{aligned} \tag{7}$$

for $x, y, z \in \mathcal{A}$.

(ii) Suppose $^-$ is the identity map and hence $\mathcal{A}$ is commutative. If $(\mathcal{A}, ^-)$ is δ-structurable then $\mathcal{A}$ is a Jordan algebra. Indeed, if $(\mathcal{A}, ^-)$ is structurable then the assertion follows from [1]§1 since char$\mathbf{F} \neq 3$. If $\mathcal{A}$ is anti-structurable we may put $x = y = h$ and $z = k$ in (7) and simplify using commutativity to obtain $h(h^2k) = h^2(hk), h, k \in \mathcal{A}$. Conversely, by [17]§3, any Jordan algebra satisfies (6) if $V_{x,y} = {}^+V_{x,y}$ for $x, y \in \mathcal{A}$, hence it is structurable. Thus, by (7), any Jordan algebra is anti-structurable if it satisfies

$$((hx)y)z - h((xy)z) = (x(yh))z - (xy)(hz), \tag{8}$$

for $h, x, y, z \in \mathcal{A}$. Using commutativity then (8) e.g. can be written $[x, h, y]z = [xy, z, h]$. Clearly, (8) is fulfilled by an associative algebra.

(iii) If $x \in \mathcal{A}$ and $T_x(1) = 0$ then $x = 0$. Indeed, if $(\mathcal{A}, ^-)$ is structurable then the assertion follows from [1]§1. If $(\mathcal{A}, ^-)$ is anti-structurable then $T_x(1) = 0$ implies, by (5), $\overline{x} = 0$ hence $x = 0$.

3. Skew-alternativity. For $s \in \mathcal{S}$ and $h \in \mathcal{H}$ we say that $(\mathcal{A}, ^-)$ is $\mathcal{S}$ *skew-alternative* if $[s, x, y] = -[x, s, y]$ while $(\mathcal{A}, ^-)$ is $\mathcal{H}$ *skew-alternative* if $[h, x, y] = -[x, h, y]$ for $x, y \in \mathcal{A}$. We shall remark that if $(\mathcal{A}, ^-)$ is $\mathcal{S}$ skew-alternative then by [1]§1,

$$[s, x, y] = -[x, s, y] = [x, y, s], s \in \mathcal{S}, x, y \in \mathcal{A},$$

while if $(\mathcal{A}, ^-)$ is $\mathcal{H}$ skew-alternative then by (1),

$$[h, x, y] = -[x, h, y] = [x, y, h], h \in \mathcal{H}, x, y \in \mathcal{A}. \tag{9}$$

Proposition 3.1. *If $(\mathcal{A}, ^-)$ is structurable, then $(\mathcal{A}, ^-)$ is $\mathcal{S}$ skew-alternative. If $(\mathcal{A}, ^-)$ is anti-structurable, then $(\mathcal{A}, ^-)$ is $\mathcal{H}$ skew-alternative.*

Proof. The first assertion follows from [1] (Proposition 1). Let now $(\mathcal{A}, ^-)$ be anti-structurable. If $h \in \mathcal{H}$ and $s \in \mathcal{S}$, putting $x = 1, y = s$ in (7) gives

$$[h, s, z] - 2[h, z, s] = [s, h, z] + [z, s, h] + [z, h, s] \tag{10}$$

for $z \in \mathcal{A}$, after changing signs. Similarly, putting $x = s, y = 1$ in (7) gives

$$[h, s, z] - 2[h, z, s] = [s, h, z] - [z, s, h] - [z, h, s]. \tag{11}$$

Subtracting (11) from (10) gives $[z, s, h] + [z, h, s] = 0$. If we apply $^-$ to this equation and substitute $z = \overline{y}$ we obtain

$$[h, s, y] = -[s, h, y], h \in \mathcal{H}, s \in \mathcal{S}, y \in \mathcal{A}. \tag{12}$$

If we put now $x = h, y = k \in \mathcal{H}$ and $z = 1$ in (7) then $3[h, h, k] = -[k, h, h]$ and so by (1), $-3[k, h, h] = [h, h, k]$. Combining these equations we obtain

$$[h, h, k] = 0, h, k \in \mathcal{H}. \tag{13}$$

Finally, putting $z = h$ in (11) we obtain $2[h, s, h] - [h, h, s] - [s, h, h] = 0$. By (12), putting $y = h$ we obtain $[h, s, h] = -[s, h, h]$ and combining with the previous identity we obtain $-3[s, h, h] = [h, h, s]$ and so by (1), $-3[h, h, s] = [s, h, h]$. Hence $9[h, h, s] = [h, h, s]$ and therefore $[h, h, s] = 0$. This combined with (13) gives $[h, h, y] = 0$ for $h \in \mathcal{H}, y \in \mathcal{A}$. If we linearize this equation, we obtain $[h, k, y] = -[k, h, y]$ for $h, k \in \mathcal{H}, y \in \mathcal{A}$, which combined with (12) implies that $(\mathcal{A}, ^-)$ is $\mathcal{H}$ skew-alternative. □

Remarks. (i) If $(\mathcal{A}, ^-)$ is anti-structurable then (9) is valid symmetrically with respect to x and y since if we put $z = 1$ in (7) then (9) implies $[x, h, y] = [y, h, x]$ for $h \in \mathcal{H}$ and $x, y \in \mathcal{A}$, since char$\mathbf{F} \neq 2$ and the assertion follows from Proposition 3.1.

(ii) Let $(\mathcal{A}, ^-)$ be a δ-structurable algebra and let $\mathrm{Der}(\mathcal{A}, ^-)$ be the set of derivations of $\mathcal{A}$ that commute with $^-$. By Remark (iii) of the previous section $T_{\mathcal{A}} \cap \mathrm{Der}(\mathcal{A}, ^-) = 0$ and so we may define the *structure algebra* $\mathrm{Str}(\mathcal{A}, ^-) := T_{\mathcal{A}} \oplus \mathrm{Der}(\mathcal{A}, ^-)$. This algebra plays an important role in the structure study of structurable algebras ([1]) and may play a role in the structure study of anti-structurable algebras (theory to be presented elsewhere).

4. (ϵ, δ)-Freudenthal Kantor triple systems and δ-Lie triple systems. Let $\epsilon, \delta = \pm 1$. Let $U(\epsilon, \delta)$ be a vector space over $\mathbf{F}$. A triple system $(x, y, z) \mapsto (xyz), x, y, z \in U(\epsilon, \delta)$, is called a (ϵ, δ)-*Freudenthal Kantor triple system* (abbreviated as (ϵ,δ)-FKTS) if the following identities are valid ([23])

(14) $$[L_{a,b}, L_{x,y}] = L_{L_{a,b}(x),y} + \epsilon L_{x,L_{b,a}(y)},$$

(15) $$K_{K_{a,b}(x),y} = L_{y,x}K_{a,b} - \epsilon K_{a,b}L_{x,y},$$

for $a,b,x,y \in U(\epsilon,\delta)$, where the endomorphisms $L_{a,b}$ and $K_{a,b}$ on $U(\epsilon,\delta)$ are defined by

(16) $$L_{a,b}(c) := (abc), \quad K_{a,b}(c) := (acb) - \delta(bca),$$

where $a,b,c \in U(\epsilon,\delta)$. A triple system satisfying only the identity (14) is called a *generalized FKTS.*

If we define

(17) $$S_{a,b} := L_{a,b} + \epsilon L_{b,a}, \quad A_{a,b} := L_{a,b} - \epsilon L_{b,a}$$

then $S_{a,b}$ is a derivation and $A_{a,b}$ is an anti-derivation of $U(\epsilon,\delta)$, by [9].

Remark. We note that a $(-1,1)$-FKTS coincides with GJTS of 2-nd order thus there can be constructed the corresponding Lie algebra ([14,15,17,22]) while for a $(-1,-1)$-FKTS there can be constructed the corresponding Lie superalgebra by the standard embedding method [5,6,9].

For $\delta = \pm 1$, a triple system $(a,b,c) \mapsto [abc]$, $a,b,c \in V$ is called a *δ-Lie triple system* if the following identities are fulfilled

$$[abc] = -\delta[bac],$$
$$[abc] + [bca] + [cab] = 0,$$
$$[ab[xyz]] = [[abx]yz] + [x[aby]z] + [xy[abz]],$$

where $a,b,x,y,z \in V$. A 1-Lie triple system is a *Lie triple system* while a -1-Lie triple system is called an *anti-Lie triple system*, by [10].

5. Lie (super)algebra construction.

Theorem 5.1. [9,11] *Let $U(\epsilon,\delta)$ be an (ϵ,δ)-FKTS. If P is an endomorphism of $U(\epsilon,\delta)$ such that $P(xyz) = (PxPyPz)$ and $P^2 = -\epsilon\delta Id$ then $(U(\epsilon,\delta), [\ \ ])$ is a Lie triple system (for $\delta = 1$) or an anti-Lie triple system (for $\delta = -1$) with respect to the product*

$$[xyz] = (xPyz) - \delta(yPxz) + \delta(xPzy) - (yPzx).$$

Corollary 5.1. [9,11] *Let $U(\epsilon,\delta)$ be an (ϵ,δ)-FKTS and the endomorphisms $L_{a,b}$ and $K_{a,b}$ be defined by* (16)*. Then the vector space $T(\epsilon,\delta) := U(\epsilon,\delta) \oplus U(\epsilon,\delta)$ is a Lie triple system (for $\delta = 1$) or an anti-Lie triple system (for $\delta = -1$) with respect to the product*

$$\left[\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}\right] = \begin{pmatrix} L_{a,d} - \delta L_{c,b} & \delta K_{a,c} \\ -\epsilon K_{b,d} & \epsilon(L_{d,a} - \delta L_{b,c}) \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}.$$

Remark. Then by [9,11], we can obtain the standard embedding Lie algebra (for $\delta = 1$) or Lie superalgebra (for $\delta = -1$) $L(\epsilon,\delta) = D(T(\epsilon,\delta), T(\epsilon,\delta)) \oplus T(\epsilon,\delta)$ associated with $T(\epsilon,\delta)$, where $D(T(\epsilon,\delta), T(\epsilon,\delta))$ is the set of inner derivations of $T(\epsilon,\delta)$, i.e.

$$D(T(\epsilon,\delta), T(\epsilon,\delta)) := \left\{ \begin{pmatrix} L_{a,b} & \delta K_{c,d} \\ -\epsilon K_{e,f} & \epsilon L_{b,a} \end{pmatrix} \right\}_{span},$$

$$T(\epsilon,\delta) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x,y \in U(\epsilon,\delta) \right\}_{span}.$$

Then by [9,11], the standard embedding Lie (super)algebra $L(\epsilon,\delta)$ is 5-graded:

$$L(\epsilon,\delta) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2, \quad [L_i, L_j] \subseteq L_{i+j},$$

such that

$$L_{-2} = \left\{ \begin{pmatrix} 0 & K_{c,d} \\ 0 & 0 \end{pmatrix} \right\}_{span}, \quad L_{-1} = U(\epsilon,\delta),$$

$$L_0 = \left\{ \begin{pmatrix} L_{a,b} & 0 \\ 0 & \epsilon L_{b,a} \end{pmatrix} \right\}_{span} = \{L_{a,b}\}_{span}.$$

Moreover $L_0 = Der\, U(\epsilon,\delta) \oplus AntiDer\, U(\epsilon,\delta)$ and $L_{-2} \oplus L_0 \oplus L_2 = D(T(\epsilon,\delta), T(\epsilon,\delta))$, where $L_{-1} \oplus L_1 = T(\epsilon,\delta) = U(\epsilon,\delta) \oplus U(\epsilon,\delta)$, by [9,11].

6. Examples.

For examples of structurable algebras we refer to [1] and [2].

Remark. Let (B,U) and (B',U') be two triple systems. We say that a linear map μ of U into U' is a *homomorphism* if μ satisfies $\mu(B(x,y,z)) = B'(\mu(x),\mu(y),\mu(z))$, $x,y,z \in U$. Moreover, if μ is bijective, then μ is called an *isomorphism.* In this case (B,U) and (B',U') are said to be *isomorphic.*

Let $(A,^-)$ be a unital non-associative algebra over $\mathbf{F}$ with involution $^-$ and let $(A^{op},^-)$ denote the *opposite* algebra, i.e. the algebra with multiplication defined by $x \cdot_{op} y = yx, x,y \in A$, where in the right hand side of the equality the multiplication is done in A. The algebras $(A,^-)$ and $(A^{op},^-)$ are isomorphic under the map $x \mapsto \overline{x}$ (this is true for any algebra with involution). Let us define

(18) $${}^{\delta}V^{op}_{x,y} := R_{R_x(\overline{y})} + \delta(L_x L_{\overline{y}} - L_y L_{\overline{x}}),$$

(19) $$^{\delta}B^{op}_{\mathcal{A}}(x,y,z) := {}^{\delta}V^{op}_{x,y}(z) = z(\overline{y}x) + \delta[x(\overline{y}z) - y(\overline{x}z)], x,y,z \in \mathcal{A}.$$

Then $\mathcal{A}$ is a δ-structurable algebra if and only if $\mathcal{A}^{op}$ is a δ-structurable algebra since clearly, $B^{op}_{\mathcal{A}}$ is the triple system obtained from the algebra $(\mathcal{A}^{op}, ^{-})$, and so $B_{\mathcal{A}}$ and $B^{op}_{\mathcal{A}}$ are isomorphic under the map $x \mapsto \overline{x}$, by (3) and (19).

Example. Let $\mathcal{M}_{m,n}(\mathbf{F})$ denote the vector space of $m \times n$ matrices over $\mathbf{F}$ and for $x \in \mathcal{M}_{m,n}(\mathbf{F})$ denote by $x^{\top}$ the transposed matrix.

(i) $\mathcal{M}_{m,n}(\mathbf{F})$ with the product

(20) $$\{x,y,z\} := xy^{\top}z + \delta(zy^{\top}x - zx^{\top}y)$$

where $x,y,z \in \mathcal{M}_{m,n}(\mathbf{F})$, is a $(-1,\delta)$-FKTS. Indeed, it is straightforward calculation to show that the identities (14) and (15) hold. Hence $\mathcal{M}_{n,n}(\mathbf{F})$ with the involution $x \mapsto x^{\top}$ is a δ-structurable algebra.

(ii) $\mathcal{M}_{m,n}(\mathbf{C})$ with the product

$$\{x,y,z\} := x\overline{y}^{\top}z + \delta(z\overline{y}^{\top}x - z\overline{x}^{\top}y)$$

where $x,y,z \in \mathcal{M}_{m,n}(\mathbf{C})$, is a $(-1,\delta)$-FKTS. Indeed, it is straightforward calculation to show that the identities (14) and (15) hold. Hence $\mathcal{M}_{n,n}(\mathbf{C})$ with the involution $x \mapsto \overline{x}^{\top}$ is a δ-structurable algebra.

Remark. By [12], the following construction of Lie superalgebras is obtained by the standard embedding method. If $U(-1,-1) := \mathcal{M}_{m,2n}(\mathbf{F})$ with the product (20) then the corresponding standard embedding Lie superalgebra is $\mathfrak{osp}(2n|2m)$ (as defined by [8]), hence the standard embedding Lie superalgebra of the anti-structurable algebra $\mathcal{M}_{2n,2n}(\mathbf{F})$ is $\mathfrak{osp}(2n|4n)$. Similarly, if $U(-1,-1) := \mathcal{M}_{m,2n+1}(\mathbf{F})$ with the product (20) then the corresponding standard embedding Lie superalgebra is $\mathfrak{osp}(2n+1|2m)$ (as defined by [8]), hence the standard embedding Lie superalgebra of the anti-structurable algebra $\mathcal{M}_{2n+1,2n+1}(\mathbf{F})$ is $\mathfrak{osp}(2n+1|4n+2)$. Furthermore, the construction of these Lie superalgebras and the correspondence with extended Dynkin diagrams will be the subject of study in a forthcoming paper. Moreover for the study of the structure theory of anti-structurable algebras the Peirce decomposition (as defined by [16]) will be considered as future work.

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