On the maximal signless Laplacian spectral radius of graphs with given matching number

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Abstract: Let $\mathcal{G}_{n,\beta}$ be the set of simple graphs of order n with given matching number β. In this paper, we investigate the maximal signless Laplacian spectral radius in $\mathcal{G}_{n,\beta}$ and characterize the extremal graphs with maximal signless Laplacian spectral radius.

Key words: Signless Laplacian; matching number; spectral radius.

1. Introduction. Let $G = G(V, E)$ be a simple graph which has no loops or multiple edges, and $V = (v_1, v_2, \cdots, v_n)$ be the set of vertices. The matrix $A(G) = (a_{ij})_{n\times n}$ is called the *adjacency matrix* of G, where $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. The polynomial $det(xI - A(G))$ is called the *characteristic polynomial of* G, denoted by $P_G(x)$. The matrix $L(G) = D(G) - A(G)$ is the Laplacian matrix of G, where $D(G) = diag(d_1, d_2, \cdots, d_n)$ is the diagonal matrix and d_i is the degree of vertex v_i. The matrix $Q(G) = D(G) + A(G)$ is called *signless Laplacian matrix* of G in [1], or *Q–matrix*. For convenience, we call it *signless Laplacian*. The eigenvalues of $Q(G)$ are denoted by $\mu_1, \mu_2, \cdots, \mu_n$. Since $Q(G)$ is a real symmetric matrix, we can order them $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. The largest eigenvalue of $A(G)$, $Q(G)$ is called the *adjacent spectral radius, the signless Lapalcian spectral radius* (*Q–spectral radius*) of G, denoted by $\rho(G)$, $\mu(G)$ respectively.

Let $X = (x_1, x_2, \cdots, x_n)$ be an eigenvector of the signless Laplacian $Q(G)$ corresponding to the eigenvalue $\mu_s, 1 \leq s \leq n$, then

$$\mu_s x_i = d_i x_i + \sum_{j\sim i} x_j, \tag{1}$$

where d_i is the degree of vertex v_i, $1 \leq i \leq n$.

Two distinct edges in a graph G are *independent* if they are not incident with a common vertex in G. A set of pairwise independent edges in G is called a *matching* in G. The *matching number* $\beta(G)$(or just β, for short) of G is the cardinality of a maximum matching of G. It is well known that $\beta(G) \leq \frac{n}{2}$ with equality if and only if G has a perfect matching. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *union* $G_1 \bigcup G_2$ is defined to be $G_1 \bigcup G_2 = (V_1 \bigcup V_2, E_1 \bigcup E_2)$. The *join* $G_1 \bigvee G_2$ of G_1 and G_2 is obtained from $G_1 \bigcup G_2$ by joining edges from each vertex of G_1 to each vertex of G_2. The *components of a graph* G are its maximal connected subgraphs. Components of odd (even) order are called the *odd* (*even*) *components*. For other notations in graph theory, we follow [2].

Recently the study of the signless Laplacian attracts some research attention. In [3], Fan *et al.* studied the signless Laplacian spectral radius of bicyclic graph with fixed order. In [4], the authors used the smallest eigenvalue of $Q(G)$ to characterize some graphs. Cvetković *et al.* gave a survey about the signless Laplacian in [5]. Some other use of the signless Laplacian can be found in [6–8].

Let $\mathcal{G}_{n,\beta}$ be the set of graphs of order n with given matching number β. In this paper we shall investigate the maximal signless Laplacian spectral radius and characterize the graphs with maximal signless Laplacian spectral radius in $\mathcal{G}_{n,\beta}$.

2. Lemmas and results. In order to get our main results, we need some technical lemmas.

Lemma 2.1 [5]. *Let G be a simple connected graph, then the largest signless Laplacian spectral radius $\mu(G)$ satisfy*

$$min\{d_i + d_j\} \leq \mu(G) \leq max\{d_i + d_j\},$$

where d_i is the degree of $v_i (i = 1, 2, \cdots, n)$. For a connected graph G, equality holds in either of these

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inequalities if and only if G is regular or semi-regular bipartite.

Lemma 2.2 [9]**.** *Suppose G is a graph on n vertices with matching β. Then there exists a set S on s vertices in G such that $G-S$ has $q=n+s-2\beta$ odd components.*

Lemma 2.3. *If G is a graph with maximal signless Laplacian spectral radius in $\mathcal{G}_{n,\beta}$. Then there exist positive odd numbers $n_1, n_2, \cdots, n_q$ such that*

$$G = K_s \bigvee \left(\bigcup_{i=1}^{q} K_{n_i}\right)$$

with $s = q+2\beta-n$ and $\sum_{i=1}^{q} n_i = n-s$.

Proof. By Lemma 2.2, there exists a subset S on s vertices in G such that $G-S$ has $q=n+s-2\beta$ odd components. Let $G_1, G_2, \cdots, G_q$ be the odd components in $G-S$ with $|V(G_i)| = n_i \geq 1$ for $i=1,2,\cdots,q$.

We claim that $G-S$ contain no even components, since G has maximal signless Laplacian spectral radius in $G_{n,\beta}$. In fact, if it does not hold, let C be the union of these even components. Then we add some edges to make $G[G_q \bigcup C]$ to be a complete graph. In this way, we get a new graph $\widetilde{G}$ and $\mu(G) < \mu(\widetilde{G})$. Moreover, $\widetilde{G}$ is a graph on n vertices with the matching number β. It is a contradiction.

Since $Q(G)$ is a real irreducible nonnegative matrix, then adding edges to G shall result in increasing $\mu(G)$. So we can have $G = K_s \bigvee (\bigcup_{i=1}^{q} K_{n_i})$. □

Lemma 2.4. *If G^* is a graph with maximal signless Laplacian spectral radius in $\mathcal{G}_{n,\beta}$. Then there exists a nonnegative number q such that*

$$G^* = K_s \bigvee (K_{n_q} \bigcup \overline{K_{q-1}}),$$
$$q = n+s-2\beta, n_q = 2\beta - 2s + 1.$$

Proof. By Lemma 2.3, a graph G with maximal signless Laplacian spectral radius should satisfy $G = K_s \bigvee (\bigcup_{i=1}^{q} K_{n_i})$ where q is a nonnegative number. Let μ be the eigenvalue of $Q(G)$, X is a eigenvector corresponding to μ. From the symmetry of vertices in K_{n_i} and K_s, we can assume the components of X corresponding to the vertices in K_{n_i} are $x_i, 1 \leq i \leq q$, the components of X corresponding to the vertices in K_s are y. By (1), we have

$$(2) \quad \begin{cases} (\mu - 2(n_1-1) - s)x_1 - sy = 0, \\ (\mu - 2(n_2-1) - s)x_2 - sy = 0, \\ \cdots\cdots\cdots \\ (\mu - 2(n_i-1) - s)x_i - sy = 0, \\ \sum_{i=1}^{q} n_i x_i - (\mu - n - s + 2)y = 0. \end{cases}$$

Let M_k be the coefficient matrix of system (2). Since $X \neq 0$, the determinant $|M_k| = 0$. By solving $|M_k|$, we get the following relation

$$|M_k| = \prod_{i=1}^{q} (\mu - 2(n_i-1) - s) \times \left[\mu - n + 2 - s - \sum_{i=1}^{q} \frac{n_i s}{\mu - 2(n_i-1) - s}\right].$$

So $\mu(G)$ satisfies

$$\mu - n + 2 - s - \sum_{i=1}^{q} \frac{n_i s}{\mu - 2(n_i-1) - s} = 0.$$

We consider the following function

$$f(\delta, \mu) = \frac{\mu - n + 2 - s}{s} - \sum_{i=1}^{q-2} \frac{n_i}{\mu - 2(n_i-1) - s} - \frac{n_{q-1} - \delta}{\mu - 2(n_{q-1} - \delta - 1) - s} - \frac{n_q + \delta}{\mu - 2(n_q + \delta - 1) - s},$$

where $\mu \geq n$ and $0 \leq \delta \leq 2$.

Taking derivative with respect to δ, we have

$$\frac{df(\delta,\mu)}{d\delta} = (\mu - s + 2) \times \frac{4(n_q - n_{q-1} + 2\delta)(n_q + n_{q-1} - \mu + s - 2)}{(\mu - 2(n_{q-1} - \delta - 1))^2 (\mu - 2(n_q + \delta - 1) - s)^2} < 0.$$

Then $f(\delta, \mu)$ is strictly decreasing with respect to δ for $\mu \geq n$.

Thus by Lemma 2.1, we have $f(2, \mu(G)) < f(0, \mu(G)) = 0$. This means that if we increase n_q by 2 and decrease n_{q-1} by 2 in G, the signless Laplacian spectral radius will increase, moreover, the resulting graph still has matching number β.

By repeating the above procedure, we can complete the proof. □

Now we present our main result.

Theorem 2.5. *Let $G \in \mathcal{G}_{n,\beta}$ be any graph on n vertices with matching number β. Then we have*

(1). *If $n = 2\beta$, or $2\beta+1$, then $\mu(G) \leq \mu(K_n)$, with equality if and only if $G \cong K_n$;*

(2). *If $2\beta + 2 \leq n < \frac{5\beta+3}{2}$, then $\mu(G) \leq 4\beta$, with* *equality if and only if* $G \cong K_{2\beta+1} \bigcup \overline{K_{n-2\beta-1}}$;

(3). *If* $n = \frac{5\beta+3}{2}$, *then* $\mu(G) \leq 4\beta$, *with equality if and only if* $G \cong K_\beta \bigvee \overline{K_{n-\beta}}$, *or* $G \cong K_{2\beta+1} \bigcup \overline{K_{n-2\beta-1}}$;

(4). *If* $n > \frac{5\beta+3}{2}$, *then* $\mu(G) \leq \frac{1}{2}(n-2+2\beta+\sqrt{(n-2+2\beta)^2 - 8\beta^2 + 8\beta})$, *with equality if and only if* $G \cong K_\beta \bigvee \overline{K_{n-\beta}}$.

Proof. From the proof of Lemma 2.4, we know that $\mu(G^*)$ satisfy $g(\mu) = 0$, where

$$\begin{aligned} g(\mu) &= (\mu - n + 2 - s)(\mu - s)(\mu - 4\beta + 3s) \\ &\quad - (n + s - 2\beta - 1)s(\mu - 4\beta + 3s) \\ &\quad - (\mu - s)s(2\beta - 2s + 1). \end{aligned}$$

It is easy to see that

$$\begin{aligned} g(s) &= 4s(\beta - s)(n + s - 2\beta - 1) \geq 0, \\ g(4\beta - 3s) &= -4s(\beta - s)(2\beta - 2s + 1) \leq 0, \\ g(+\infty) &> 0, \\ g(-\infty) &< 0. \end{aligned}$$

Hence the three roots of $g(\mu) = 0$ lie in three intervals $(-\infty, s)$, $(s, 4\beta - 3s)$, $(4\beta - 3s, +\infty)$. So we conclude that $g(\mu) = 0$ has exactly one root $\geq 4\beta - 3s$.

(1). If $n = 2\beta$, or $2\beta + 1$, it is easy to know that $\mu(G) \leq \mu(K_n)$ with equality if and only if $G \cong K_n$.

(2). If $2\beta + 2 < n < \frac{5\beta+3}{2}$, by Lemma 2.4, we need just to verify that $\mu(G^*) \leq \mu(H)$, where $H = K_\beta \bigvee \overline{K_{n-\beta}}$. A direct computation shows that $\mu(H)$ satisfy $h(\mu) = 0$, where

$$h(\mu) = \mu^2 - (n - 2 + 2\beta)\mu + 2\beta^2 - 2\beta.$$

Moreover, if $n < \frac{5\beta+3}{2}$, $\mu(H) < \mu(K_{2\beta+1} \bigcup \overline{K_{n-2\beta-1}}) = 4\beta$.

A direct computation shows that

$$\begin{aligned} g(\mu) &= (\mu - 4\beta)(\mu^2 + (-n + 2 + s)\mu \\ &\quad + s(12\beta - 3n - 4s + 4)) \\ &\quad + 2s(20\beta^2 + 10\beta - 4s\beta - s - s^2 - 6n\beta). \end{aligned}$$

So we can easily verify

$$\begin{aligned} g(4\beta) &= 2s(20\beta^2 + 10\beta - 4s\beta - s - s^2 - 6n\beta) \\ &\geq 2s(20\beta^2 + 10\beta - 4s\beta - s - s^2 - 15\beta^2 - 9\beta) \\ &= 2s(5\beta^2 + \beta - 4s\beta - s - s^2) \\ &= 2s(\beta - s)(5\beta + s + 1) \geq 0. \end{aligned}$$

This means that $\mu(G^*) \leq 4\beta$. If $\mu(G^*) = 4\beta$, then $s = 0$. From Lemma 2.4, we have $G^* \cong H$.

(3). If $n = \frac{5\beta+3}{2}$, we have $g(4\beta) = 2s(\beta - s)(5\beta + s + 1) \geq 0$, hence, $\mu(G^*) \leq 4\beta$.

If $\mu(G^*) = 4\beta$, then $s = 0$, or $\beta = s$, which implies our result.

(4). If $n > \frac{5\beta+3}{2}$, from the proof of (1), it is easy to see that $\mu(H)$ satisfies

$$h(\mu) = \mu^2 - (n - 2 + 2\beta)\mu + 2\beta^2 - 2\beta = 0,$$

where $H = K_\beta \bigvee \overline{K_{n-\beta}}$. Moreover, we know that

$$\begin{aligned} \mu(H) &= \frac{1}{2}(n - 2 + 2\beta) \\ &\quad + \sqrt{(n - 2 + 2\beta)^2 - 8\beta^2 + 8\beta} > 4\beta. \end{aligned}$$

So we have

$$\begin{aligned} g(\mu) &= h(\mu)(\mu - 2\beta + s) \\ &\quad + (\beta - s)(2n - 2 + 4s - 6\beta)\mu \\ &\quad + (\beta - s)(2s - 6s\beta - 4\beta + 4\beta^2 + 2s^2). \end{aligned}$$

Hence we can verify

$$\begin{aligned} g(\mu(H)) &= (\beta - s)(2n - 2 + 4s - 6\beta)\mu(H) \\ &\quad + (\beta - s)(2s - 6s\beta - 4\beta + 4\beta^2 + 2s^2) \\ &\geq (\beta - s)[(2n - 2 + 4s - 6\beta)4\beta + 2s \\ &\quad - 6s\beta - 4\beta + 4\beta^2 + 2s^2] \\ &\geq (\beta - s)[(5\beta + 3 - 2 + 4s - 6\beta)4\beta + 2s \\ &\quad - 6s\beta - 4\beta + 4\beta^2 + 2s^2] \\ &= (\beta - s)(10s\beta + 2s + 2s^2) \\ &= 2s(\beta - s)(5\beta + s + 1) \\ &\geq 0. \end{aligned}$$

This means that $\mu(G^*) \leq \mu(H)$.

If $\mu(G^*) = \mu(H)$, then $\beta = s$, which implies our result. □

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