On an analog of Serre's conjectures, Galois cohomology and defining equation of unipotent algebraic groups

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Abstract: In this note we establish the validity, in the case of unipotent group schemes over non-perfect fields, of an analog of Serre's conjectures for algebraic groups, which relates properties of Galois (or flat) cohomology of unipotent group schemes to finite extensions of non-perfect fields. We also establish an interesting property of Russell's defining equations of connected smooth one-dimensional unipotent groups over a field k.

Key words: Serre's conjectures; unipotent groups; Galois cohomology.

Introduction. If G is a smooth (i.e., absolutely reduced) affine group scheme defined over a perfect field k then one may define its first Galois cohomology $\mathrm{H}^1(k,G) := \mathrm{H}^1(\mathrm{Gal}(k_s/k), G(k_s))$, where $\mathrm{Gal}(k_s/k)$ denotes the absolute Galois group of k. In [Se1; Chap. III, Sec. 2.2, Sec. 2.3 and Sec. 3.1] Serre formulated his famous conjectures I and II (see also [Se2; Sec. 4 and 5]). Recall them briefly (cf. [Se2; pp. 236, 237]), as follows:

(I) *Let k be a perfect field. Then*

$$cd(k) \leq 1 \text{ if and only if } \mathrm{H}^1(k,G) = 0$$

for all connected smooth k-groups G.

(This is now celebrated Steinberg's Theorem [St]; it was extended by Borel - Springer [BS] to the case of arbitrary (not necessarily perfect) fields while restricting to connected reductive groups only).

(II) *Let k be a perfect field. Then*

$$cd(k) \leq 2 \text{ if and only if } \mathrm{H}^1(k,G) = 0$$

for all semisimple simply connected k-groups G.

We refer to [Se2,BaP1,BaP2,BFT,Gi], for more recent results in the direction of Serre's conjecture (II), the general case of which is still open. In (I) if one drops the condition of perfectness of the field, one needs to restrict oneself to the case of connected reductive groups. It is due to the fact (see, e.g. [Se1]), that if G is a smooth connected (resp. and unipotent) group defined over a perfect field k then its unipotent radical is defined over k (resp. its first Galois cohomology $\mathrm{H}^1(k,G)$ is trivial), but these facts are no longer true if we drop the perfectness condition on k. In fact, even over some fields, such as global (resp. local) function fields, every smooth unipotent groups of dimension one which is not isomorphic to $\mathbf{G}_a$ (the additive group) (resp. if $char(k)$ is not 2) has infinite Galois cohomology (see, [TT]).

In the first section of this paper, for any field k and for any unipotent k-group scheme G, we prove the existence of a normal composition series $G = G_0 > G_1 > \cdots > G_n = \{1\}$ of k-subgroup schemes of G, such that G_{i+1} is normal and of codimension 1 in G_i, for all $i \geq 0$. Then, by using Whaples' methods [W1,W2,W3], we propose a necessary and sufficient condition on k which ensures the triviality of the first Galois cohomology set for all smooth unipotent groups defined over k. One of the characterizations is the statement, which is an analog of Serre's characterization of cohomological dimension of the base field, via the triviality of the first Galois cohomology of algebraic groups (see (I), (II) above). We establish an analog of Serre's conjectures for unipotent group schemes via Theorems 3 and 6. Then using this, we describe relations between various statements regarding finite extensions of degree p or divisible by p of a given field k (Theorem 9). In the second section, we study the

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equations defining smooth connected unipotent groups of dimension one. In [Ru], P. Russell shows that every smooth connected unipotent k-groups of dimension 1 is k-isomorphic to a k-subgroup of $\mathbf{G}_a^2$ defined by a p-polynomial of the form

$$(1) \qquad F(x,y) := y^{p^n} - (x + a_1x^p + \cdots + a_rx^{p^r}),$$

where some $a_i \notin k^p$. This explicit equation proves to be of great importance in the study of arithmetic of unipotent group schemes over fields and rings (cf. e.g. [KMT,Oe,WW]). In [KMT], many results in [Ru] have been generalized. For example, it has been shown that the number n in (1) is uniquely determined by G (see Corollary of Theorem 2.4.3, [KMT]). Using results of [KMT], we show further that the set $\{i \mid a_i \notin k^p\}$ also depends only on G. In fact, we prove a slightly more general result (see Proposition 10).

We recall some basic definitions about the theory of unipotent groups over fields (see, e.g. [KMT,Oe,Ti,SGA3; Exp. XVII]). The smooth affine algebraic groups considered here are the same as linear algebraic groups in the sense of [Bo]. Let k be a field. An affine algebraic group scheme defined over k is called *unipotent* if it is k-isomorphic to a closed k-subgroup scheme of the matrix group consisting of all upper triangular matrices with all 1 on their main diagonal (see [SGA3; Exp. XVII]). For simplicity, we call smooth unipotent k-group schemes just *unipotent k-groups.* We recall after Tits that a unipotent k-group scheme G is called *k-wound* if every k-homomorphism (or even k-morphism) $\mathbf{G}_a \to G$ is constant. A polynomial $P := P(x_1, \ldots, x_n)$ in n variable $x_1, \ldots, x_n$ with coefficients in k is said to be *universal* if $P(k^n) = k$. We say that P is *additive* if $P(x+y) = P(x) + P(y)$, for any two elements $x \in k^n, y \in k^n$, where $k^n := k \times \cdots \times k$ (n times). If this is the case, P is the so-called *p-polynomial*, i.e, a k-linear combination of $x_i^{p^{m_{i,j}}}$. Denote by $\mathrm{H}^1_{fppf}(k, G)$ the flat cohomology of G.

1. Triviality of the first Galois cohomology group. In this section, we first show by using Tits results [Ti], that for all unipotent k-group schemes G of dimension n, there exists a composition series of normal k-subgroup schemes $G_n = G > G_{n-1} > \cdots > G_1 > G_0 = \{1\}$ such that $\dim G_i = i$. In fact, this fact is already implicitly contained in [Ti], (and in [Ke] one may find another proof in the case of smooth connected group schemes). Then, we give equivalent conditions for a non-perfect field k which are sufficient and necessary for the triviality of the first Galois cohomology of an arbitrary smooth unipotent group defined over k.

Proposition 1. *Let G be a unipotent k-group scheme of dimension ≥ 1. Then there exists a normal k-subgroup scheme G' of codimension* 1 *in G. If, moreover, G is smooth* (*resp. connected, resp. connected and smooth*)*, G' can be chosen smooth* (*resp. connected, resp. connected and smooth*)*, too.*

Remarks. 1) We can choose G' even connected. Indeed, if G' was chosen, then it is clear that its connected component $(G')^\circ$ is also normal in G and has codimension 1 there.
2) This proposition is useful in the regard, that one may use induction on the dimension and dévissage, i.e., one may consider the initial unipotent group scheme as an extension of two other groups, of smaller dimension, or order, etc..., to study various properties of unipotent group schemes over fields.

Corollary 2. *Let G be a unipotent k-group scheme of dimension n. Then there is a composition series of normal k-subgroup schemes $G_n = G > G_{n-1} > \cdots > G_1 > G_0 = \{1\}$ such that $\dim G_i = i$. If moreover G is smooth* (*resp. connected, resp. connected and smooth*)*, the k-subgroup schemes G_i can be chosen smooth* (*resp. connected, resp. connected and smooth*)*, too.*

Proof. Use induction on dimension of G. □

Remarks. If G is a k-wound unipotent group scheme, then one can show [TT1] that it has a composition series of characteristic k-wound k-subgroups with commutative k-wound factors of exponent p, but one cannot expect in general G to have a composition series such that all factors are *one-dimensional* and also k-wound. In fact, assume that over any non-perfect field of characteristic $p > 0$ every k-wound unipotent group G had a composition series such that all factors were also k-wound and one-dimensional. Then it also holds for global function fields. By an induction argument and using [Oe; Théorème 3.1, p. 65], we can show that for all k-wound unipotent groups G defined over the global function field k, if the above assumption were true, it would imply that $G(k)$ were finite. But it would contradict Example 3.4, p. 68 of Oesterlé [Oe], which shows that there exists a k-wound unipotent group G of dimension $p-1$ such that $G(k)$ is infinite, where k can be any global function field of characteristic $p > 2$.

Theorem 3. *Let k be an arbitrary field of characteristic $p > 0$. The following statements are equivalent*:

1) *k has no Galois extensions of degree divisible by p*;
2) *k has no separable extensions of degree divisible by p*;
3) *Every separable p-polynomial in one variable is universal*;
4) $\mathrm{H}^1(k, G) = 0$ *for all smooth unipotent k-groups G.*

Remarks. 1) Recall that a field k is called *Kaplansky field* if every p-polynomial is universal (see [Va,W2]). By Theorem 1 of [W2], k is Kaplansky field if and only if it has no finite extensions of degree divisible by p. So this theorem can be considered as an analog of Theorem 1 of Whaples [W2].

2) In [Ru], P. Russell remarks that if k has no normal extension of degree $p = char(k)$ then $\mathrm{H}^1(k, G)$ is trivial for all smooth connected unipotent k-group G of dimension 1 (and then, as one sees below, the same is true for smooth connected unipotent groups of arbitrary dimension). In fact, the conclusion is trivially true since any non-perfect field k always has a normal extension of degree $p = char(k)$, thus, if k has no normal extension of degree p, then k is perfect. However, notice that if a field k has no normal extensions of degree p then it is not necessarily true that every separable p-polynomial in one variable is universal (condition 3 in Theorem 3). For, if this were true then the condition that "k has no normal extensions of degree p" would imply the condition 1) in Theorem 3, but it would contradict a Whaples' result which is stated as follows:

Theorem 4 (Whaples [W1]). *Let n be any positive integer. There exists a field K which has algebraic extensions of degrees divisible by n but has no extensions of degree $\leq n$.*

In fact, the proof of the above theorem even shows that, we can choose K such that K has Galois extensions of degrees divisible by n, but no extensions of degree $\leq n$.

3) Let $\mathcal{G}$ be a profinite group, p a prime number. We recall that (see [Se1; Chapter I, Sec.3]) *p-cohomological dimension* of $\mathcal{G}$, denoted by $cd_p(\mathcal{G})$, is the lower bound of the integers n such that for every discrete torsion $\mathcal{G}$-module A, and for every $q > n$, the p-primary component of $\mathrm{H}^q(\mathcal{G}, A)$ is null. One defines (as in [Se1; Chap. I, Sec. 1, 1.3]) *supernatural numbers* as formal products $\prod_p p^{n_p}$, where p runs over the set of all prime numbers, and n_p is either a non-negative integer or ∞. One then defines the product, greatest common divisor (g.c.d), least common multiple (l.c.m.) of a family of supernatural numbers in an obvious manner (loc.cit). For a profinite group G and its closed subgroup H we define the index $[G : H]$ as l.c.m. of all indices $\{[(G/U) : (H/(H \cap U)]\}$, where U runs over all open normal subgroups of G. In particular, the order of G is $[G : \{1\}]$. We have the following proposition.

Proposition 5 ([Se, Chap. I, Sec.3.3, Corol. 2]). *In order that $cd_p(\mathcal{G}) = 0$ it is necessary and sufficient that the order of $\mathcal{G}$ be prime to p.*

On the other hand, by Galois theory, k has no Galois extensions with Galois groups of (supernatural) order divisible by p if and only if the Galois group $\mathcal{G} = Gal(k_s/k)$ has the (supernatural) order prime to p. Hence, we can restate a part of Theorem 3 in cohomological terms as an analog of (or a complement to) Serre's conjectures (I) and (II) as follows:

Theorem 6 (Analog of Serre's conjectures for unipotent group schemes). *Let k be a field of characteristic $p > 0$ and let $cd_p(k) := cd_p(Gal(k_s/k))$ be the cohomological p-dimension of k. Then*

a) $cd_p(k) = 0$ if and only if $\mathrm{H}^1_{fppf}(k, G) = 0$ for all smooth unipotent k-groups G.

b) k is perfect and $cd_p(k) = 0$ if and only if $\mathrm{H}^1_{fppf}(k, G) = 0$ for all unipotent k-group schemes G.

Proof (Sketch). We need only prove b), since part a) follows from above. By considering the infinitesimal k-group scheme α_p represented by the k-algebra $k[T]/T^p$, the direction ($\Leftarrow$) is clear. For the direction ($\Rightarrow$), we need only show that $\mathrm{H}^1_{fppf}(k, G) = 0$ for all infinitesimal k-group schemes G. Any such a group scheme G has a central composition series

$$G = G_0 > G_1 > \cdots > G_n = \{1\},$$

where each successive quotient $G_i/G_{i+1} \simeq \alpha_p$ for all $0 \leq i \leq n-1$ (see [SGA3; Exp. XVII, Théorème 3.5]). Since $\mathrm{H}^1_{fppf}(k, \alpha_p) = k/k^p = 0$, it follows by dévissage that the same holds for G. □

Next we deduce from above some corollaries. We give one example of non-perfect fields k satisfying equivalent conditions of Theorem 3. We have the following

Corollary 7. *There are non-perfect non-separably closed fields k such that for all smooth unipotent groups G over k we have $\mathrm{H}^1(k, G) = 0$.*

Proof. Indeed, let k_0 be an arbitrary non-perfect non-separably closed field of characteristic p, such that its absolute Galois group $\mathcal{G} := Gal(k_{0,s}/k_0)$ is not a p-group, i.e., it has order >not of the form p^α, where $\alpha \leq \infty$ and that there exists a prime $q \neq p$ dividing the order of $\mathcal{G}$, and $\mathcal{G}$ is not a q-group. (For example, one may take p odd, $k_0 = \mathbf{F}_p(t)$ the rational function field in one variable over $\mathbf{F}_p$, and $q = 2$.) Thus its q-Sylow subgroup S_q is non-trivial and $S_q \neq \mathcal{G}$. Take $k = k_{0,s}^{S_q}$, the fixed field of S_q in $k_{0,s}$. Then $Gal(k_{0,s}/k) = S_q$ is a non-trivial pro-q-group and k is a desired field. For, it follows from Theorem 3, Proposition 5 and Theorem 6 above, that every separable p-polynomial in one variable with coefficients in k is universal over k. Now we prove that k is non-perfect by showing that $t \notin k^p$, for any $t \in k_0 - k_0^p$. Otherwise, let $t = \alpha^p \in k^p$. In the tower of extension fields $k_0 \subset k_0(\alpha) \subset k$, $k_0(\alpha)/k_0$ is purely inseparable extension and also separable since k/k_0 is separable. Thus $k_0(\alpha) = k_0$, and $t = \alpha^p \in k_0^p$, a contradiction. Also, k is non-separably closed, since $S_q \neq \{1\}$. $\square$

Corollary 8. *Let k be an arbitrary field of characteristic $p > 0$ and let k' be a finite extension of k. Then, $\mathrm{H}^1(k, G) = 0$ for all smooth unipotent k-groups G if and only if $\mathrm{H}^1(k', G') = 0$ for all smooth unipotent k'-groups G'.*

The proof follows from Theorems 3, 6 and from [Se1; Chap. II, Prop. 10]. From above we derive the following second main result of this section.

Theorem 9. *Let k be any field of characteristic $p > 0$. Consider the following statements*

1) *k has no extensions of degree p;*
2) *k has no extensions of degree divisible by p;*
3) *k has no normal extensions of degree p;*
4) *k has no normal extensions of degree divisible by p;*
5) *k has no Galois extensions of degree p;*
6) *k has no Galois extensions of degree divisible by p;*
7) *Every separable p-polynomial in one variable is universal;*
8) *Every p-polynomial in one variable is universal.*
9) $\mathrm{H}^1(k, G) = 0$ *for any smooth unipotent k-group G.*

Then we have the following diagram of relations

$$\begin{array}{ccccc} 1) & \Rightarrow & 3) & \Rightarrow & 5) \\ \Uparrow & & \Uparrow & & \Uparrow \\ 2) & \Leftrightarrow & 4) & \Rightarrow & 6) \\ \Updownarrow & & & & \Updownarrow \\ 8) & \Rightarrow & 7) & \Leftrightarrow & 9) \end{array} \qquad \begin{array}{ccccc} 1) & \not\Leftarrow & 3) & \not\Leftarrow & 5) \\ \not\Downarrow & & \not\Downarrow & & \not\Downarrow \\ 2) & \Leftrightarrow & 4) & \not\Leftarrow & 6) \\ \Updownarrow & & & & \Updownarrow \\ 8) & \not\Leftarrow & 7) & \Leftrightarrow & 9) \end{array}$$

All other related implications or non-implications between the statements above follow from these diagrams.

Remarks. 1) One might add the 10-th condition, saying that $\mathrm{H}^1_{fppf}(k, G) = 0$ for all unipotent k-group schemes G, which is equivalent to conditions 2) and 8), but it is a bit difficult to draw the square above of relations with ten vertices.

2) From above (Theorem 3 and Corollary 7) we see that the two conditions in Theorem 6 are not the same.

3) We give an example, which shows that the condition

"$\mathrm{H}^1(k, G) = 0$ *for all smooth unipotent k-groups G*"

and the condition

"$\mathrm{H}^1(k, G) = 0$ *for all connected and smooth unipotent k-groups G*"

are not the same. Indeed, take any field k of characteristic $p > 0$ such that certain separable p-polynomial $f(T)$ in one variable with coefficients in k (e.g. the Artin - Schreier map $\wp$) is not surjective as a map $k^+ \to k^+$. Take $a \in k^+ \setminus f(k^+)$. We claim that for the perfect closure $K := k^{-p^\infty}$ of k, we have $K^+ \neq f(K^+)$. If not, $K^+ = f(K^+)$, and we have $a \in f(K^+)$, so $a = f(x)$, $x \in K$. By assumption, we have $k \neq k(x)$. Since x is a root of the separable polynomial $f(T) - a$, $k(x)/k$ is a separable extension. But $k(x) \subset K$ and K/k is purely inseparable, hence so is $k(x)/k$. Thus $k = k(x)$, which is impossible. Therefore $f(K^+) \neq K^+$. Let $G := Ker(f)$. Then G is a finite (smooth) étale unipotent k-group scheme with $\mathrm{H}^1(K, G) = K^+/f(K^+) \neq 0$, while $\mathrm{H}^1(K, H) = 0$ for all connected smooth unipotent k-groups H, since K is perfect.

2. Equations defining commutative unipotent groups of exponent p. We first recall some notations and results in Part 1 of [KMT]. Let k be a non-perfect field of characteristic $p > 0$. It is known that the endomorphism ring $R := End_{k-gr}(\mathbf{G}_a)$ can be identified with the noncommutative polynomial k-algebra with one indeterminate F subjected to the relation $F\lambda = \lambda^p F$, for all $\lambda \in k$. A pair (n, α) with $n \in \mathbf{N}$ and $\alpha = \sum a_i F^i \in k[F]$ is called *admissible* if either (i) $n = 0$ or (ii) $a_0 \neq 0$ and $a_i \notin k^p$ for some $i > 0$. For a commutative affine k-group scheme G, denote $M(G) := \mathrm{Hom}_{k-gr}(G, \mathbf{G}_a)$, which is a left R-module, hence also a left $k[F]$-module in a natural way. On the other hand, any given left $k[F]$-module M can be considered as a (commutative) p-Lie algebra with zero multiplication and p-power given by $m^{[p]} = Fm$, for all $m \in M$. The universal envelopping k-algebra $U(M)$ of M has a natural Hopf algebra structure, and the affine k-group scheme corresponding to $U(M)$ is denoted by $D(M)$. It has been shown that (cf. [DG; Chap. IV, Sec. 3, no. 6.2, p. 520]) there is an anti-equivalence between the category of commutative k-group schemes with the category of left $k[F]$-modules, via $G \mapsto M(G)$; $M \mapsto D(M)$, where the algebraic k-group schemes correspond to finitely generated modules.

Let $M(n, \alpha)$ be the left $k[F]$-module on a set of 2 generators x, y defined by the relation $F^n y = \alpha x$, $\alpha \in k[F]$. Then, there is a natural bijective correspondence $G \mapsto M(G)$; $M \mapsto D(M)$ between the unipotent groups G of dimension 1 and left $k[F]$-modules $M = M(n, \alpha)$, where (n, α) are admissible pairs. More precisely, if G is defined by the equation $y^{p^n} = a_0 x + a_1 x^p + \cdots + a_r x^{p^r}$, where $a_0 \neq 0$ and $a_i \notin k^p$ for some i, then to G one assigns $M(G) := M(n, \alpha)$, where $\alpha = a_0 + a_1 F + \cdots + a_r F^r$. For $\alpha = \sum a_i F^i \in k[F]$, let $\alpha^{(\nu)} = \sum a_i^{p^\nu} F^i$.

Proposition 10. *Let k be a non-perfect field of characteristic $p > 0$, and let G_1, G_2 be unipotent smooth k-groups of dimension 1, defined by*
$\{(x, y) \in \mathbf{G}_a^2 | y^{p^m} = x + a_1 x + \cdots + a_r x^{p^r}, \exists i, a_i \notin k^p\}$,
$\{(x, y) \in \mathbf{G}_a^2 | y^{p^n} = x + b_1 x + \cdots + b_s x^{p^s}, \exists j, b_j \notin k^p\}$
respectively. If the groups $\mathrm{Hom}_{k-gr}(G_1, G_2)$ *and* $\mathrm{Hom}_{k-gr}(G_2, G_1)$ *are both nontrivial then $m = n$ and the following two sets of indices coincide*:

$$\{i \mid a_i \notin k^p\} \equiv \{i \mid b_i \notin k^p\}.$$

Proof (Sketch). We first prove the second statement of Proposition 10 in the particular case when $n = m = 1$.

Lemma 11. *Let k be a non-perfect field of characteristic $p > 0$, and let G_1, G_2 be unipotent smooth k-groups of dimension 1, defined by*
$\{(x, y) \in \mathbf{G}_a^2 | y^p = x + a_1 x + \cdots + a_r x^{p^r}, \exists i, a_i \notin k^p\}$,
$\{(x, y) \in \mathbf{G}_a^2 | y^p = x + b_1 x + \cdots + b_s x^{p^s}, \exists j, b_j \notin k^p\}$
respectively. Assume that $\mathrm{Hom}_{k-gr}(G_1, G_2)$ *and* $\mathrm{Hom}_{k-gr}(G_2, G_1)$ *are nontrivial. Then we have*

$$\{i \mid a_i \notin k^p\} \equiv \{i \mid b_i \notin k^p\}.$$

Now, we proceed to prove the proposition. Let

$$\alpha = 1 + a_1 F + \cdots + a_r F^r, \beta = 1 + b_1 F + \cdots + b_s F^s.$$

Then we have $G_1 \simeq D(M(m, \alpha)), G_2 \simeq D(M(n, \beta))$, where $(m, \alpha), (n, \beta)$ are admissible pairs and by assumption, $\mathrm{Hom}_{k[F]}(M(m, \alpha), M(n, \beta))$ and $\mathrm{Hom}_{k[F]}(M(n, \beta), M(m, \alpha))$ are nontrivial. Assume that $n > m$. Then by [KMT], Theorem 5.3.1, we have $\mathrm{Hom}_{k[F]}(M(m, \alpha), M(n, \beta)) = 0$. So $n \leq m$. Similarly, $m \leq n$ and we get $n = m$. Let $G_1' = G_1^{(p^{n-1})}$ and $G_2' = G_2^{(p^{n-1})}$. Then from [KMT], Proposition 5.4.1 it follows that $\mathrm{Hom}_{k-gr}(G_1', G_2')$ and $\mathrm{Hom}_{k-gr}(G_2', G_1')$ are nontrivial. Let $r' = \max\{i \mid a_i \notin k^p\}, s' = \max\{j : b_j \notin k^p\}$. Thus we have $G_1' \simeq D(M(1, \alpha'))$, $G_2' \simeq D(M(1, \beta'))$, where $\alpha = 1 + a_1 F + \cdots + a_{r'} F^{r'}$, $\beta = 1 + b_1 F + \cdots + b_{s'} F^{s'}$. By Lemma 11, we get $\{i \mid a_i \notin k^p\} \equiv \{i \mid b_i \notin k^p\}$ and Proposition 10 follows □

Corollary 12. *Let k be a non-perfect field of characteristic $p > 0$, and let G be an unipotent k-group of dimension* 1*. If there are k-isomorphisms of G with the following k-groups*
$\{(x, y) \in \mathbf{G}_a^2 | y^{p^m} = x + a_1 x^p + \cdots + a_r x^{p^r}, \exists i, a_i \notin k^p\}$
and
$\{(x, y) \in \mathbf{G}_a^2 | y^{p^n} = x + b_1 x^p + \cdots + b_s x^{p^s}, \exists j, b_j \notin k^p\}$.
then $m = n$ and $\{i \mid a_i \notin k^p\} \equiv \{i \mid b_i \notin k^p\}$.

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