Getzler's relation by localization

By Fumitoshi SATO

School of Mathematics, Korea Institute for Advanced Study,
207-43 Cheongnyangni 2-dong, Dongdaemun-gu, Seoul 130-722, Republic Korea

(Communicated by Shigefumi MORI, M.J.A., Oct. 12, 2007)

Abstract: We obtain the Getzler's relation in $H^4(\overline{M}_{2,2})$ by the localization theorem, which will give a constructive proof of Getzler's conjecture, that is, the relation is actually algebraic.

Key words: Moduli of curves; Localization theorem.

1. Introduction. Let $M_{g,n}$ be the moduli of genus g smooth curves with n distinct marked points defined over the complex numbers. There is a compactification of $M_{g,n}$ denoted by $\overline{M}_{g,n}$ which is the moduli of genus g stable curves with n marked points. A genus g stable curve with n marked points is an arithmetic genus g complete connected nodal curve with distinct smooth n marked points and finite automorphisms. $\overline{M}_{g,n}$ has a stratification according to topological types, which are organized by the dual graphs. The vertices of a dual graph correspond to irreducible components, the edges correspond to intersections of irreducible components, and the half-edges correspond to marked points. In the case of genus 2, the irreducible components all have genus at most 2. Following Getzler [4], we will represent a genus 0 component by a point, a genus 1 component by a circle, a genus 2 component by a circled 2.

There are several natural vector bundles on $\overline{M}_{g,n}$. The ψ-classes and λ-classes are defined using Chern classes of these vector bundles.

Consider the forgetting map π, that forgets the last marked point,

$$\begin{tikzcd} \overline{M}_{g,n+1} \arrow[d, bend left, "\pi"] \\ \overline{M}_{g,n}. \arrow[u, bend left, "s_i"] \end{tikzcd}$$

The $\overline{M}_{g,n+1}$ is the universal curve of $\overline{M}_{g,n}$ and there are n-sections $s_1, \cdots, s_n$. The ψ-classes are defined as follows:

$$\psi_i := c_1(s_i^*(\omega_{\overline{M}_{g,n+1}/\overline{M}_{g,n}}))$$

where $\omega_{\overline{M}_{g,n+1}/\overline{M}_{g,n}}$ is the relative dualizing sheaf.

The Hodge bundle $\mathbf{E}$ is defined by

$$\mathbf{E} := \pi_* \omega_{\overline{M}_{g,n+1}/\overline{M}_{g,n}}.$$

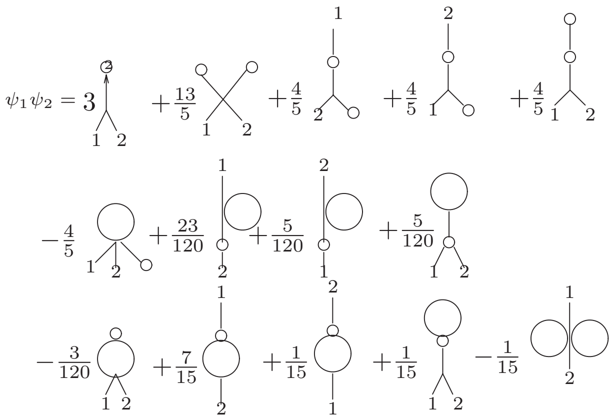
It is a rank g bundle on $\overline{M}_{g,n}$ such that

$$\mathbf{E}|_C = H^0(\omega_C)$$

for $C \in \overline{M}_{g,n}$. The λ-classes are defined as follows,

$$\lambda_i := c_i(\mathbf{E}).$$

Getzler proved that



in $H^4(\overline{M}_{2,2})$ by computing all the betti numbers and intersection matrix of $H^*(\overline{M}_{2,2})$, and conjectured his relation is actually a rational equivalence relation [4]. We prove this is the case.

One theoretical proof of the Geztler's conjecture was given by T. Graber and R. Vakil [6].

2. Background.

2.1. Dimension formula. In this section, we list dimensions for the stacks which we will deal with.

Let $\overline{M}_{g,n}(X, \beta)$ be the moduli of genus g, n-pointed stable maps to a projective variety X with

2000 Mathematics Subject Classification. Primary 14N10, 14N35.

β class. A genus g, n-pointed stable map to X with β class is a map from a genus g nodal curve with n smooth distinct points to X whose image has class $\beta \in A_1(X)$ [3]. By deformation theory and the Riemann-Roch formula,

$$\text{vir. dim } \overline{M}_{g,n}(X,\beta) = (1-g)(dim X - 3) - \int_\beta \omega_X + n.$$

The special cases which we will use frequently in this paper are:

$$\text{vir. dim. } \overline{M}_{g,n} = \text{dim. } \overline{M}_{g,n} = 3g-3+n$$
$$\text{vir. dim. } \overline{M}_{g,n}(\mathbf{P}^1, d) = 2g-2+2d+n.$$

Remark 1. *If $X = point$, then $\overline{M}_{g,n}(X,\beta)$ is just $\overline{M}_{g,n}$.*

2.2. Virtual localization. The higher genus Kontsevich–Manin spaces $\overline{M}_{g,n}(\mathbf{P}^m, d)$ are in general non-reduced, reducible, singular, so we can not apply the localization formula. The answer to overcome this difficulty is the virtual localization theorem by Graber and Pandharipande [5].

Theorem 2 (The virtual localization theorem [5]). *Suppose $f : X \to X'$ is a $T = (C^*)^{m+1}$-equivariant map of proper Deligne–Mumford quotient stacks with a T-equivariant perfect obstruction theory. If $i' : F' \hookrightarrow X'$ is a fixed substack and $c \in A_T^*(X)$, let $f_{F_i} : F_i \to F'$ be the restriction of f to each of the fixed substacks $F_i \subset f^{-1}(F')$. Then*

$$\sum_{F_i} f_{F_i *} \frac{i_{F_i}^* c}{\epsilon_T(F_i^{vir})} = \frac{i'^* f_* c}{\epsilon_T(F'^{vir})}$$

where $i_{F_i} : F_i \hookrightarrow X$ and $\epsilon_T(F^{vir})$ is the virtual equivariant Euler class of "virtual" normal bundle F^{vir}.

Remark 3.

(a) *If X and X' are nonsingular with the trivial perfect obstruction theories* [1], *then the virtual localization formula reduces to the standard localization formula.*

(b) *The conditions in the theorem are satisfied for the Kontsevich–Manin spaces $\overline{M}_{g,n}(\mathbf{P}^m, d)$, and $\epsilon_T(F^{vir})$ can be explicitly computed in terms of ψ and λ-classes* [5].

2.3. $\mathbf{C}^*$-action on $\mathbf{P}^1$. We define a $T = \mathbf{C}^*$-action on $\mathbf{P}^1$ for $a \in T$ and $(x_0 : x_1) \in \mathbf{P}^1$ by

$$a \cdot (x_0 : x_1) = (x_0 : a x_1).$$

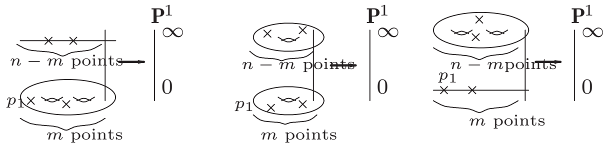


Fig. 1. General Fixed Loci.

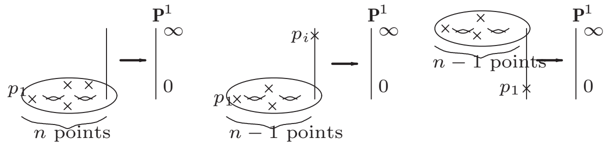


Fig. 2. Three Types of Degeneration.

There are two fixed points $\infty = (0:1)$ and $0 = (1:0)$.

This T-action induces T-actions on $\overline{M}_{g,n}(\mathbf{P}^1, d)$.

3. Pushforward of $\boldsymbol{\psi_{n+1}^k}$. Consider the following map

$$f : \overline{M}_{2,n}(\mathbf{P}^1, 1) \to \overline{M}_{2,n} \times \mathbf{P}^1;$$
$$\Big(g : (C; p_1, \cdots, p_n) \to X\Big) \mapsto \Big((C; p_1, \cdots, p_n)^{\mathrm{st}}, g(p_1)\Big)$$

where $(C; p_1, \cdots, p_n)^{\mathrm{st}}$ is the "stabilized" curve obtained by contracting all the rational components which have at most two special points.

We know that $f_*(1) = c_0$ by dimension counting. If we choose the fixed locus $F' = \overline{M}_{2,n} \times 0 (\cong \overline{M}_{2,n})$ in the target space, then $\epsilon_T(F') = t$ so

$$\sum_{F_i} f_{F_i *} \frac{i_{F_i}^* 1}{\epsilon_T(F_i)} = \frac{c_0}{t}. \tag{1}$$

Now, we have to write up all the components of fixed loci in the domain which maps to F'. The general fixed loci are as in Figure 1 (these are boundary divisors). There are three types of degeneration (Figure 2). The first one is the case all the points are on the genus 2 curve over 0, which is isomorphic to $\overline{M}_{2,n+1}$. The second and third ones have only one point on a rational tail. These are isomorphic to $\overline{M}_{2,n}$.

By the virtual localization theorem [5], we get

$$\pi_*\left(\frac{\lambda_2 - \lambda_1 t + t^2}{t(t-\psi_{n+1})}\right) + \sum_{i=2}^n \left(\frac{\lambda_2 - \lambda_1 t + t^2}{t(t-\psi_i)(-t)}\right) + \left(\frac{\lambda_2 + \lambda_1 t + t^2}{(-t)(-t-\psi_1)t}\right) \tag{2}$$

$$+\sum_{1\in I,|I^c|\geq 2}\iota_*\left(\frac{\lambda_2-\lambda_1 t+t^2}{t(t-\psi_{|I|+1})}\frac{1}{(-t)(-t-\psi_{|I^c|+1})}\right)$$
$$+\sum_{1\in I,|I|\geq 2}\iota_*\left(\frac{1}{t(t-\psi_{|I|+1})}\frac{\lambda_2+\lambda_1 t+t^2}{(-t)(-t-\psi_{|I^c|+1})}\right)$$
$$+\sum_{1\in I}\iota_*\left(\frac{-\lambda_1+t}{t(t-\psi_{|I|+1})}\frac{-\lambda_1-t}{(-t)(-t-\psi_{|I^c|+1})}\right)=\frac{c_0}{t}$$

where $\pi:\overline{M}_{2,n+1}\to\overline{M}_{2,n+1}$, $\iota:\overline{M}_{i,m+1}\times\overline{M}_{2-i,n-m+1}\to\overline{M}_{2,n}$ which corresponds to a general fixed locus in Figure 1 and $I\subset\{1,2,\cdots,n\}$.

To compute $\pi_*(\lambda_2\psi_{n+1}^{k-2}-\lambda_1\psi_{n+1}^{k-1}+\psi_{n+1}^{k})$, calculate the coefficient of t^{-k}. We know that $\pi_*(\lambda_i\psi_{n+1}^{k-1})=\lambda_i\pi_*\psi_{n+1}^{k-1}$, so we can compute $\pi_*\psi_{n+1}^{k}$ inductively.

Example 4.

(a) If we take the coefficient of t^0, only the first term will contribute, so $\pi_*1=0$.

(b) If we take the coefficient of t^{-1}, the last three summations will not contribute. We will obtain $\pi_*(-\lambda_1+\psi_{n+1})-(n-1)+1=c_0$ and this will imply that $\pi_*\psi_{n+1}=n-2+c_0$. We know, however $\pi_*(\psi_{n+1})=n+2$ [7], so $c_0=4$. (In fact, we can prove the statement by the localization theorem. [8]).

(c) If we take the coefficient of t^{-2}, we will obtain $\pi_*(\lambda_2-\lambda_1\psi_{n+1}+\psi_{n+1}^2)=\sum_{i=1}^{n}(-\lambda_1+\psi_i)\pm\Delta$ where $\pm\Delta$ is the all the boundary divisors with appropriate signs.

4. Getzler's relation by localization. In this section, we will obtain Getzler's relations by the localization calculation, that implies Getzler's relation is actually a rational equivalence relation not only a homological equivalence.

Consider the following map

$$\overline{M}_{2,3}(\mathbf{P}^1,2)\to\overline{M}_{2,3}\times\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1;$$
$$\left(g:(C;p_1,p_2,p_3)\to\mathbf{P}^1\right)$$
$$\mapsto\left((C;p_1,p_2,p_3)^{\mathrm{st}},g(p_1),g(p_2),g(p_3)\right). \tag{3}$$

Take $\overline{M}_{2,3}\times 0\times\infty\times\infty$ as a fixed locus in the target space. There are 20 fixed loci in the domain which map to $\overline{M}_{2,3}\times 0\times\infty\times\infty$ (Figure 3).

By the localization theorem with $c=1$ and by taking the coefficient of t^{-5}, we can obtain a relation in $A^2(\overline{M}_{2,3})$, i.e. $\sum_{i=1}^{20}$ (the coefficitent of t^{-5} of contribution of i-th fixed locus in Figure 3) $=0$. Multiply this relation by ψ_3 and pushforward to $\overline{M}_{2,2}$ to obtain $6\times$ Getzler's relation.

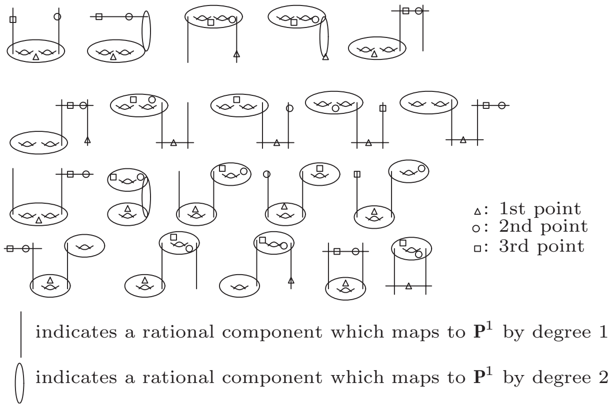
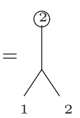


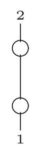
Fig. 3. 20 Fixed Loci.

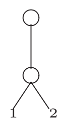
Contribution of the first fixed locus is following

$$\begin{aligned}
\text{The 1st term} &= \pi_*((\psi_3^2+\psi_3(-\lambda_1+\psi_2)\\
&\quad+\lambda_2-\lambda_1\psi_2+\psi_2^2)\cdot\psi_3)\\
&=\pi_*(\psi_3^3-\lambda_1\psi_3^2+\lambda_2\psi_3)\\
&\quad+\pi_*(\psi_2\psi_3^2-\lambda_1\psi_2\psi_3)+\pi_*(\psi_2^2\psi_3)\\
&=-\lambda_1\psi_2+\psi_2^2+\lambda_1\psi_1-\psi_1^2\\
&\quad+\iota_*^{\delta_2}(-\lambda_1{}'+\psi_1{}')+\iota_*^{\delta_{1,2}}(-\lambda_1{}'+\psi_3{}')\\
&\quad+\iota_*^{\delta_{1,1}}(-\lambda_1{}'+\psi_2{}'+\lambda_1{}''-\psi_2{}'')-2\lambda_1\psi_2+\psi_2^2\\
&\quad+\psi_1\psi_2+\psi_2(\delta_{1,1}+\delta_{1,2})+4\psi_2^2\\
&=-3\lambda_1\psi_2+6\psi_2^2+\lambda_1\psi_1-\psi_1^2\\
&\quad+\psi_1\psi_2+\iota_*^{\delta_2}(-\lambda_1{}'+\psi_1{}')\\
&\quad+\iota_*^{\delta_{1,2}}(-\lambda_1{}'+\psi_2{}'+\psi_3{}')\\
&\quad+\iota_*^{\delta_{1,1}}(-\lambda_1{}'+\psi_2{}'+\lambda_1{}'')
\end{aligned}\tag{4}$$

where

$\delta_2=$ 

$\delta_{1,1}=$ 

$\delta_{1,2}=$ 

and ι_*^{δ} indicates pushforward from δ to $\overline{M}_{2,2}$.

Remark 5. *We use computation in* §3 *repeatedly to compute* $\pi_*(p(\lambda,\psi)\cap\psi_{n+1}^k)$ *and well-known relations on the smaller genus moduli to simplify.*

Other terms can be calculated by the same way.

Acknowledgments. This paper is part of my thesis at the University of Utah. The author is grateful to Prof. A. Bertram, his thesis adviser for help in mathematics and also encouragement. The author also thanks Dr. K. Chu.

References

[1] K. Behrend and B. Fantechi, The intrinsic normal cone, Invent. Math. **128** (1997), no. 1, 45–88.

[2] D. Edidin and W. Graham, Localization in equivariant intersection theory and the Bott residue formula, Amer. J. Math. **120** (1998), no. 3, 619–636.

[3] W. Fulton and R. Pandharipande, Notes on stable maps and quantum cohomology, in *Algebraic geometry—Santa Cruz* 1995, 45–96, Proc. Sympos. Pure Math., Part 2, Amer. Math. Soc., Providence.

[4] E. Getzler, Topological recursion relations in genus 2, in *Integrable systems and algebraic geometry* (*Kobe/Kyoto,* 1997), 73–106, World Sci. Publ., River Edge, NJ.

[5] T. Graber and R. Pandharipande, Localization of virtual classes, Invent. Math. **135** (1999), no. 2, 487–518.

[6] T. Graber and R. Vakil, Relative virtual localization and vanishing of tautological classes on moduli spaces of curves, Duke Math. J. **130** (2005), no. 1, 1–37.

[7] J. Kock, Notes on Psi Classes. (Unpublished).

[8] F. Sato, Relations in tautological rings by localization, Ph. D thesis, University of Utah.