A note on the 3-class field tower of a cyclic cubic field

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Abstract: The present note studies a sufficient condition that the length of the 3-class field tower of a cyclic cubic field is greater than 1.

Key words: Class field tower; Hilbert class field; class number; cyclic cubic field.

1. Introduction. For a number field K and a prime p, we denote by the sequence

$$K = K_0^{(p)} \subseteq K_1^{(p)} \subseteq K_2^{(p)} \subseteq \cdots$$

the p-class field tower of K, that is, $K_{n+1}^{(p)}$ is the Hilbert p-class field of $K_n^{(p)}$. We define an invariant $l_p(K)$ called the "length of the Hilbert p-class field tower of K" as follows: $l_p(K)$ is the smallest non-negative integer i such that $K_i^{(p)} = K_{i+1}^{(p)}$, if such an integer exists, and $l_p(K)$ is ∞ otherwise. Replacing "Hilbert p-class field" by "Hilbert class field" in the above, we define an invariant $l(K)$ called the "length of the Hilbert class field tower of K".

Golod-Shafarevich[6] proved that there exist infinitely many number fields K such that $l(K) = \infty$. Hajir[7] has determined all imaginary quadratic fields K such that $l(K) = 1$, and Benjamin-Lemmermeyer-Snyader[1] has determined all real quadratic fields K such that $l_2(K) = 1$. In case K is a certain biquadratic field, Yoshida[11] studied the condition for $l_3(K) > 1$. In this paper we shall study the case where K is a cyclic cubic field, and give the condition for $l_3(K) > 1$.

2. Theorem and the proof. Let K be a number field and p a prime. Let $Cl_K(p)$ be the p-Sylow subgroup of the ideal class group of K and $\rho_K(p)$ the rank of E/E^p where E denotes the unit group of K. For a Galois extension $K/\mathbf{Q}$, we denote by $m(K)$ the number of rational primes ramified in $K/\mathbf{Q}$. Bond[2] proved the following

Lemma 1 (Bond [2, Corollary 3.10])**.** *If the rank of $Cl_K(p)$ is greater than $\frac{1+\sqrt{1+8\rho_K(p)}}{2}$, then $l_p(K) > 1$.*

Remark. An alternative proof of this lemma is found in Nomura[8, Corollary 4]. The approach is based on the theory of embedding problems.

By using Lemma 1 and the genus theory, we get the following lemma.

Lemma 2. *Let K be a cyclic cubic field. If the 3-rank of $Cl_K(3)$ is greater than or equal to 3, then $l_3(K) > 1$. In particular, if $m(K) > 3$, then $l_3(K) > 1$.*

We shall study the case of $m(K) = 3$ and get the following theorem.

Theorem 3. *Let K be a cyclic cubic field such that $m(K) = 3$. Then the following conditions (1) and (2) are equivalent.*

(1) $l_3(K) > 1$.

(2) *The class number of the genus field of K is divisible by 3.*

We need the following lemma, which is easily proved. For the proof, see for example [9, Lemma 7.2].

Lemma 4. *Let p be an odd prime and $F/\mathbf{Q}$ a p-extension. If the class number of F is divisible by p, then there exists a Galois extension $L/\mathbf{Q}$ such that L contains F, L/F is unramified, and the degree $[L : F]$ is equal to p.*

Proof of Theorem 3. (1) $\Rightarrow$ (2) is trivial. We shall consider (2) $\Rightarrow$ (1). Let K_g be the genus field of K. Then $K_g/\mathbf{Q}$ is a Galois extension and $\mathrm{Gal}(K_g/\mathbf{Q}) \cong \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$. Assume that the class number of K_g is divisible by 3. By Lemma 4, there exists a Galois extension $L/\mathbf{Q}$ such that L contains K_g, L/K_g is unramified, and the degree $[L : K_g]$ is equal to 3. Let $G = \mathrm{Gal}(L/\mathbf{Q})$. Then the group G satisfies the conditions:

(a) the order of G is 3^4;

(b) the rank of $G/[G, G]$ is 3.

By virtue of Tchebotareff's monodoromy

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theorem[3, Theorem 16.30], we see that the group G also satisfies the condition:

(c) G is generated by the elements of order 3.

Let Γ be the group defined by

$$\Gamma = \left\langle x, y, z \,\middle|\, \begin{array}{l} x^3 = y^3 = z^3 = 1 \\ x^{-1}yx = xz, xz = zx, yz = zy \end{array} \right\rangle.$$

By using GAP[5], it is easy to see that the group satisfying the conditions (a) (b) and (c) is isomorphic to $\Gamma \times \mathbf{Z}/3\mathbf{Z}$. See also Schreier[10] for the groups of order p^3, p^4, p^5. Then G is isomorphic to $\Gamma \times \mathbf{Z}/3\mathbf{Z}$. Further we can check that any maximal subgroup of G is isomorphic to Γ or $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$.

Since $\mathrm{Gal}(L/K)$ is isomorphic to a maximal subgroup of G, it is enough to consider the following two cases.

Case 1. $\mathrm{Gal}(L/K) \cong \Gamma$: Since Γ is a non-abelian 3-group, $l_3(K) > 1$.

Case 2. $\mathrm{Gal}(L/K) \cong \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$: Since the 3-rank of $Cl_K(3)$ is greater than or equal to 3, $l_3(K) > 1$ by Lemma 2.

We have thus proved our theorem.

Remark. Assume that the prime 3 is unramified in $K/\mathbf{Q}$. Then the condition (2) in Theorem 3 is easily checked by using the result in Cornell-Rosen[4, Section 3].

Remark. In the previous work Nomura[9, Corollary 7.4], we have proved the following. Assume that K is a cyclic cubic field such that $m(K) = 2$ and that the prime 3 is unramified in $K/\mathbf{Q}$. If the class number of K is divisible by 27, then $l_3(K) > 1$.

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