On reachability of parallel-flow heat exchanger equations with boundary inputs

By Hideki Sano

Computing and Communications Center, Kagoshima University, 1–21–35 Korimoto, Kagoshima 890–0065, Japan

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Abstract: This paper is concerned with the problem of reachability of parallel-flow heat exchanger equations with boundary inputs. It is shown that the system with boundary inputs is formulated as a boundary control system which is well-defined in the sense of Curtain and Zwart (1995), and further that it is reachable through a concrete expression of the solution. In addition, the reachable subspace is given for the case where only one boundary input is added to the system.

Key words: Parallel-flow heat exchanger equation; boundary input; reachability; C_0-semigroup; characteristic differential equation.

1. Introduction. In this paper, we shall consider the following type of parallel-flow two-fluid heat exchanger equation [8]:

$$\begin{aligned} &\frac{\partial z_1}{\partial t}(t,x) = -\frac{\partial z_1}{\partial x}(t,x) + h_1(z_2(t,x) - z_1(t,x)),\\ &\frac{\partial z_2}{\partial t}(t,x) = -\frac{\partial z_2}{\partial x}(t,x) + h_2(z_1(t,x) - z_2(t,x)),\\ &\qquad\qquad (t,x) \in (0,\infty)\times[0,1],\\ &z_1(t,0) = u_1(t), \quad z_2(t,0) = u_2(t), \quad t \in (0,\infty),\\ &z_1(0,x) = z_{10}(x), \quad z_2(0,x) = z_{20}(x), \quad x \in [0,1], \end{aligned} \tag{1}$$

where $z_1(t,x)$, $z_2(t,x) \in \mathbf{R}$ are the temperature variations at time t and at the point $x \in [0,1]$ with respect to an equilibrium point, $u_1(t)$, $u_2(t) \in \mathbf{R}$ are the control inputs, and h_1, h_2 are positive physical parameters. This type of heat exchanger equations has been treated in the fields of machine industry, chemical industry and the like (see [1] and the references therein).

In Takahashi [8], the transfer function approach was adopted to analyze heat exchange processes such as parallel and counter flow types. After that, for counter-flow heat exchange processes, transient solutions were analytically derived by Jaswon and Smith [4] and Malinowski and Bielski [6]. On the other hand, for parallel-flow heat exchange processes containing system (1), exact transient solutions were concretely given by Li [5]. However, as far as the author knows, it has not been reported on reachability of these systems, although they are described by simple equations. So, in our recent study [7], we treated the reachability problem of system (1) and showed that the approximated system is reachable and that it is reachable with respect to the nonnegative cone X_+ in the state space X when $u_2 \equiv 0$, using a C_0-semigroup generated by the adjoint operator of the state operator. In this paper, we discuss on reachability of the original system with boundary inputs without approximating it.

2. Boundary control system. In this section, we shall show that system (1) defines a boundary control system

$$\frac{dz(t)}{dt} = \mathfrak{A}z(t), \;\; z(0) = z_0, \quad \mathfrak{B}z(t) = u(t) \tag{2}$$

in the sense of Curtain and Zwart [2, Definition 3.3.2]. For this purpose, we first introduce a Hilbert space $X := [L^2(0,1)]^2$ with inner product defined by

$$\langle f, g\rangle_X := \int_0^1 \{f_1(x)g_1(x) + f_2(x)g_2(x)\}dx, \qquad f = [f_1, f_2]^T \in X, \; g = [g_1, g_2]^T \in X.$$

Defining the unbounded operator $A : D(A) \subset X \to X$ as

$$\begin{aligned} &(Af)(x) = \begin{bmatrix} -\frac{d}{dx} - h_1 & h_1 \\ h_2 & -\frac{d}{dx} - h_2 \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}, \qquad f = [f_1, f_2]^T \in D(A),\\ &D(A) = \{\, f = [f_1, f_2]^T \in [H^1(0,1)]^2\,; \; f_1(0) = 0, \; f_2(0) = 0 \,\}, \end{aligned} \tag{3}$$

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the operator A generates a C_0-semigroup e^{tA} on X as follows [7]:

- In the case of $x < t$

$$\left(e^{tA}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right)(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4}$$

- In the case of $x \geq t$

$$\left(e^{tA}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right)(x) = \frac{1}{2}\begin{bmatrix} \phi_{11}(t)z_1(x-t) + \phi_{12}(t)z_2(x-t) \\ \phi_{21}(t)z_1(x-t) + \phi_{22}(t)z_2(x-t) \end{bmatrix}, \tag{5}$$

where

$$\phi_{11}(t) := 1 + e^{-(h_1+h_2)t} + \frac{h_2 - h_1}{h_1 + h_2}(1 - e^{-(h_1+h_2)t}),$$
$$\phi_{12}(t) := \frac{2h_1}{h_1 + h_2}(1 - e^{-(h_1+h_2)t}),$$
$$\phi_{21}(t) := \frac{2h_2}{h_1 + h_2}(1 - e^{-(h_1+h_2)t}),$$
$$\phi_{22}(t) := 1 + e^{-(h_1+h_2)t} - \frac{h_2 - h_1}{h_1 + h_2}(1 - e^{-(h_1+h_2)t}).$$

From (4) and (5), it is clear that the C_0-semigroup e^{tA} has the property $e^{tA} = 0\,(t \geq 1)$, where e^{tA} is said to be a nilpotent semigroup [3]. Therefore, system (1) is super-stable without adding any control inputs since the growth bound of e^{tA} is $-\infty$.

Now, let us introduce the operator $\mathfrak{A} : D(\mathfrak{A}) \subset X \to X$

$$\mathfrak{A}f = \begin{bmatrix} -\frac{df_1}{dx} + h_1(f_2 - f_1) \\ -\frac{df_2}{dx} + h_2(f_1 - f_2) \end{bmatrix}, \quad f = [f_1, f_2]^T \in D(\mathfrak{A}) = [H^1(0,1)]^2,$$

and the boundary operator $\mathfrak{B} : D(\mathfrak{B}) \subset X \to \mathbf{R}^2$

$$\mathfrak{B}f = [f_1(0), f_2(0)]^T, \quad f = [f_1, f_2]^T \in D(\mathfrak{B}) = D(\mathfrak{A}).$$

Then, the operator $\mathfrak{A}$ whose domain is restricted to $D(\mathfrak{A}) \cap \ker \mathfrak{B}$ is equal to A. Therefore, it generates a C_0-semigroup. Here, defining the bounded operator $B \in \mathcal{L}(\mathbf{R}^2, X)$ as

$$Bu = \begin{bmatrix} (1-x)u_1 \\ (1-x)u_2 \end{bmatrix}, \quad u = [u_1, u_2]^T \in \mathbf{R}^2,$$

it follows that $Bu \in D(\mathfrak{A}) = D(\mathfrak{B})$ for each $u \in \mathbf{R}^2$, and that

$$\mathfrak{B}Bu = \mathfrak{B}\begin{bmatrix} (1-x)u_1 \\ (1-x)u_2 \end{bmatrix} = \begin{bmatrix} (1-0)u_1 \\ (1-0)u_2 \end{bmatrix} = u, \; u \in \mathbf{R}^2.$$

Moreover, noting that

$$\mathfrak{A}Bu = \mathfrak{A}\begin{bmatrix} (1-x)u_1 \\ (1-x)u_2 \end{bmatrix} = \begin{bmatrix} u_1 + h_1(1-x)(u_2 - u_1) \\ u_2 + h_2(1-x)(u_1 - u_2) \end{bmatrix} \in X, \quad u \in \mathbf{R}^2,$$

it follows that $\mathfrak{A}B \in \mathcal{L}(\mathbf{R}^2, X)$. In this way, we see that system (1) defines a boundary control system in the sense of [2, Definition 3.3.2].

By [2, Chapter 3], the reachable subspace of system (2) is expressed as

$$R_b = \left\{ f \in X\,;\, \exists \tau > 0,\, \exists u \in H^1(0,\tau;\mathbf{R}^2) \text{ s.t. } u(0) = 0,\; f = Bu(\tau) + \int_0^\tau e^{(\tau-s)A}\mathfrak{A}Bu(s)ds - \int_0^\tau e^{(\tau-s)A}B\dot{u}(s)ds \right\}^a, \tag{6}$$

where $\{\cdots\}^a$ denotes the closure of the set $\{\cdots\}$. Here, note that system (2) (i.e., system (1)) is called reachable if $R_b = X$.

3. Main results. Since it is difficult to solve the reachable subspace of system (2) concretely based on the abstract expression (6), we directly calculate, in this paper, the solution of system (1) according to the method of characteristic differential equation, and then state the results concerning reachability.

First, defining

$$f(t,x) := z_1(t,x) - z_2(t,x), \tag{7}$$

we have the following equation from system (1):

$$\frac{\partial f}{\partial t}(t,x) + \frac{\partial f}{\partial x}(t,x) = -(h_1 + h_2)f(t,x), \quad (t,x) \in (0,\infty) \times [0,1],$$
$$f(t,0) = a(t), \quad t \in (0,\infty),$$

where $a(t) := u_1(t) - u_2(t)$. Then, the characteristic differential equation is given by

$$\frac{dt}{1} = \frac{dx}{1} = \frac{df}{-(h_1+h_2)f}.$$

By solving this, we get

$$f(t,x) = a(t-x)e^{-(h_1+h_2)x}. \tag{8}$$

Nextly, defining

$$g(t,x) := h_2 z_1(t,x) + h_1 z_2(t,x), \tag{9}$$

we have the following equation from system (1):

$$\begin{aligned} &\frac{\partial g}{\partial t}(t,x) + \frac{\partial g}{\partial x}(t,x) = 0, \\ &\qquad (t,x) \in (0,\infty) \times [0,1], \\ &g(t,0) = b(t), \quad t \in (0,\infty), \end{aligned}$$

where $b(t) := h_2 u_1(t) + h_1 u_2(t)$. Then, similarly as in the above, we get

$$g(t,x) = b(t-x). \tag{10}$$

Here, combining (7), (8), (9), and (10), we get

$$\begin{aligned} z_1(t,x) &= \frac{h_2 + h_1 e^{-(h_1+h_2)x}}{h_1+h_2} u_1(t-x) \\ &\quad + \frac{h_1 - h_1 e^{-(h_1+h_2)x}}{h_1+h_2} u_2(t-x), \\ z_2(t,x) &= \frac{h_2 - h_2 e^{-(h_1+h_2)x}}{h_1+h_2} u_1(t-x) \\ &\quad + \frac{h_1 + h_2 e^{-(h_1+h_2)x}}{h_1+h_2} u_2(t-x). \end{aligned} \tag{11}$$

Then, we have the following theorem:

Theorem 3.1. *System* (2) (*i.e., system* (1)) *is reachable, in other words,* $R_b = X$.

Proof. We have only to show that, for each $[\tilde{z}_1, \tilde{z}_2]^T \in [H^1(0,1)]^2$ and for any fixed $\tau > 1$, there exists $u \in H^1(0,\tau;\mathbf{R}^2)$, $u(0) = 0$ such that

$$z_1(\tau,x) = \tilde{z}_1(x), \quad z_2(\tau,x) = \tilde{z}_2(x), \tag{12}$$

because $[H^1(0,1)]^2$ is dense in X. It is easy to see that (12) is equivalent to

$$\begin{aligned} u_1(\tau-x) &= \frac{h_2 + h_1 e^{(h_1+h_2)x}}{h_1+h_2} \tilde{z}_1(x) \\ &\quad + \frac{h_1 - h_1 e^{(h_1+h_2)x}}{h_1+h_2} \tilde{z}_2(x), \\ u_2(\tau-x) &= \frac{h_2 - h_2 e^{(h_1+h_2)x}}{h_1+h_2} \tilde{z}_1(x) \\ &\quad + \frac{h_1 + h_2 e^{(h_1+h_2)x}}{h_1+h_2} \tilde{z}_2(x). \end{aligned}$$

For each $i = 1,2$, let $\zeta_i \in H^1(0,\tau)$ be an extension of $\tilde{z}_i \in H^1(0,1)$ such that

$$\zeta_i|_{[0,1]} = \tilde{z}_i, \quad \zeta_i(\tau) = 0.$$

Here, setting the inputs as

$$\begin{aligned} u_1(t) &= \frac{h_2 + h_1 e^{(h_1+h_2)(\tau-t)}}{h_1+h_2} \zeta_1(\tau-t) \\ &\quad + \frac{h_1 - h_1 e^{(h_1+h_2)(\tau-t)}}{h_1+h_2} \zeta_2(\tau-t), \\ u_2(t) &= \frac{h_2 - h_2 e^{(h_1+h_2)(\tau-t)}}{h_1+h_2} \zeta_1(\tau-t) \\ &\quad + \frac{h_1 + h_2 e^{(h_1+h_2)(\tau-t)}}{h_1+h_2} \zeta_2(\tau-t), \end{aligned} \tag{13}$$

it follows that $u_1 \in H^1(0,\tau;\mathbf{R})$, $u_1(0) = 0$ and $u_2 \in H^1(0,\tau;\mathbf{R})$, $u_2(0) = 0$. Furthermore, we see that (12) is satisfied with the inputs $u_1(t)$, $u_2(t)$ given by (13). □

Finally, we shall give the reachable subspace of the system with only one boundary input. In the case of $u_2 \equiv 0$, the reachable subspace of the system is expressed as

$$\begin{aligned} \hat{R}_b = \Big\{ f \in X\,;\, &\exists\, \tau > 0,\, \exists\, u_1 \in H^1(0,\tau;\mathbf{R}) \quad \text{s.t.} \\ &u_1(0) = 0, \\ &f = \hat{B}u_1(\tau) + \int_0^\tau e^{(\tau-s)A} \hat{\mathfrak{A}} \hat{B} u_1(s)\,ds \\ &\qquad - \int_0^\tau e^{(\tau-s)A} \hat{B} \dot{u}_1(s)\,ds \Big\}^a, \end{aligned}$$

where the operator A is defined by (3), and the other operators are defined by

$$\hat{\mathfrak{A}} f = \begin{bmatrix} -\frac{df_1}{dx} + h_1(f_2 - f_1) \\ -\frac{df_2}{dx} + h_2(f_1 - f_2) \end{bmatrix}, \quad f = [f_1, f_2]^T \in D(\hat{\mathfrak{A}}),$$

$$D(\hat{\mathfrak{A}}) = \{\, f = [f_1, f_2]^T \in [H^1(0,1)]^2\,;\, f_2(0) = 0 \,\},$$

and

$$\hat{B} u_1 = \begin{bmatrix} (1-x)u_1 \\ 0 \end{bmatrix}, \quad u_1 \in \mathbf{R}.$$

Theorem 3.2. *Consider the system* (1) *with* $u_2 \equiv 0$. *Then, there holds*

$$\begin{aligned} \hat{R}_b = \Big\{ \big[\, f,\, \frac{h_2 - h_2 e^{-(h_1+h_2)(\cdot)}}{h_2 + h_1 e^{-(h_1+h_2)(\cdot)}} f \,\big]^T \in X\,; \\ f \in L^2(0,1) \Big\}. \end{aligned} \tag{14}$$

Proof. When $u_2 \equiv 0$, it follows from (11) that

$$z_1(t,x) = \frac{h_2 + h_1 e^{-(h_1+h_2)x}}{h_1+h_2} u_1(t-x), \tag{15}$$
$$z_2(t,x) = \frac{h_2 - h_2 e^{-(h_1+h_2)x}}{h_1+h_2} u_1(t-x).$$

By the similar discussion as in the proof of Theorem 3.1, it is possible to show that, for each $\tilde{z}_1 \in H^1(0,1)$ and for any fixed $\tau > 1$, there exists $u_1 \in H^1(0,\tau;\mathbf{R})$, $u_1(0) = 0$ such that $z_1(\tau,x) = \tilde{z}_1(x)$. Here, putting $t = \tau$ in (15) and eliminating u_1 yields

$$z_2(\tau,x) = \frac{h_2 - h_2 e^{-(h_1+h_2)x}}{h_2 + h_1 e^{-(h_1+h_2)x}} \tilde{z}_1(x).$$

Finally, noting that $H^1(0,1)$ is dense in $L^2(0,1)$, we have (14). □

Remark 3.1. The solution (15) to the system (1) with $u_2 \equiv 0$ has been given in [5]. Note that, in the paper, the method by the Laplace tranformation is used.

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