Compactifications of the iso level sets of the Hessenberg matrices and the full Kostant-Toda lattice

By Kaoru IKEDA

Mathematics, Hiyoshi Campus, Keio University, Hiyoshi 4–1–1,
Kouhoku-ku, Yokohama 223–8521, Japan

Abstract: We consider the problem of the compactification of the iso level sets of the Hessenberg matrices which is propounded by Ercolani et al. We determine the structure of the cohomology ring of the compact iso level set and obtain a new expression of the flag variety G/B.

Key words: Full Kostant-Toda lattice; Gauss decomposition; iso level set; flag manifold.

1. Introduction. In this note, we study the compactification of the iso level set of the Hessenberg matrices. Let Λ be a shift matrix $\sum_{i=1}^{n-1} E_{i,i+1}$. The matrices of the form $\Lambda + \bar{\mathtt{b}}$ are called Hessenberg matrices, where $\bar{\mathtt{b}}$ is the Borel subalgebra of the lower triangular matrices. The equation of the Hessenberg matrix $L(t)$, $\dot{L}(t) = [L(t)_+, L(t)]$ is called the full Kostant-Toda lattice, where $L(t)_+$ is the projection of $L(t)$ on the Borel subalgebra of upper triangular matrices. The orbit of the full Kostant-Toda lattice $L(t)$ stays in a given iso level set. However, $L(t)$ has poles in finite time. By adding points at infinity to the iso level set, the compactification of the iso level set is obtained. In the case of the ordinary Toda lattice, the compactifiction of the iso level set is already studied by Flaschka and Haine[2]. By Painlevé analysis for singularities of the Toda lattice, they characterize the poles by elements of Weyl group. They realize the compact iso level set of the Toda lattice as the toroidal orbit in the flag variety G/B. Since the Lax operator of the ordinary Toda lattice is a tri-diagonal matrix($2n$-dimensional system), we can regard their compactification as a certain variant of the Theorem of Liouville and Arnol'd. In the case of the full Kostant-Toda lattice, the degree of freedom is so many $(n(n+1)/2)$ compared with the ordinary Toda lattice. Ercolani et al. consider the geometry of the full Kostant-Toda lattice by using new integrals (k-chop integrals)[1]. We parameterize poles of the full Kostant-Toda lattice by points of the another algebraic variety. By adding the poles to the iso level set, we obtain new variety. The cell decom-

position of the new variety implies the compactness of it and also determines **Z**-module structure of the cohomology group. Moreover, considering the line bundles defined by eigen vectors of the Hessnberg matrices, we obtain the structure of its cohomology ring. The cohomology ring of the compact iso level set is isomorphic to that of the flag variety G/B. This fact asserts the new expression of the flag variety by using iso level set. If we restrict eigenvalues $\alpha = (\alpha_1, \ldots, \alpha_n)$(see the following section) to $\mathbf{R}^n$, the compact iso level set in this note remains in real. However we can not treat the structure of the cohomology of the compact iso level set as complex case. If we find a metric so that the orbits of the full Kostant-Toda lattice are geodesics on the iso level set of the full Kostant-Toda lattice, the full Kostant-Toda flows would be applied to the conic compactification[3] of the real iso level sets of Hessenberg matrices.

2. Iso spectral varieties in the Lie algebra. Let G be $GL(n, \mathbf{C})$, B a Borel subgroup of upper triangular matrices and N a unipotent subgroup of B. Furthermore $\bar{B}$ and $\bar{N}$ are the opposites of B and N respectively. Let $\mathtt{g}$, $\mathtt{b}$, $\mathtt{n}$, $\bar{\mathtt{b}}$ and $\bar{\mathtt{n}}$ be the Lie algebras of G, B, N, $\bar{B}$ and $\bar{N}$ respectively. Let us provide $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{C}^n$. We assume that $\alpha_i \neq \alpha_j$ if $i \neq j$. The iso spectral set $\mathcal{A}_\alpha$ is the subset of $\mathtt{g}$ whose eigenvalues are $\alpha_1, \ldots, \alpha_n$. Note that $\mathcal{A}_\alpha$ is $n^2 - n$ dimensional algebraic variety. Its defining equations are $|\alpha_k I_n - X| = 0$, $k = 1, \ldots, n$. Let $P_{i,j}(X, \alpha_k)$ be the i, j cofactor of $|\alpha_k I_n - X|$. Fix i, then we have the following equations by expansion along the i-th row of $|\alpha_k I_n - X|$

$$(1) \qquad P_{i,1}(X, \alpha_k)x_{i,1} + \cdots + P_{i,n}(X, \alpha_k)x_{i,n}$$

$$= \alpha_k P_{i,i}(X, \alpha_k), \ k = 1, \ldots, n.$$

Put $Q_i(X, \alpha) = \det(P_{i,j}(X, \alpha_k))$. The i-th coordinate neighborhood $\mathcal{V}_i$ is defined by $\{X \in \mathcal{A}_\alpha | Q_i(X, \alpha) \neq 0\}$. The defining equations become $x_{i,j} = Q_{i,j}(X, \alpha)/Q_i(X, \alpha), j = 1, \ldots, n$ on $\mathcal{V}_i$, where $Q_{i,j}(X, \alpha)$ is the determinant of matrix whose j-th colum is ${}^t(\alpha_1 P_{i,i}(X, \alpha_1), \ldots, \alpha_n P_{i,i}(X, \alpha_n))$ and the other colums are the same as those of $Q_i(X, \alpha)$. Let $\mathcal{S}_n$ be the n-th symmetric group. Let $\epsilon > 0$ be a sufficiently small number. For $A = (a_{i,j}) \in \mathsf{g}$, we define the norm $||A||$ by $||A|| = \max_{1 \leq i,j \leq n} |a_{i,j}|$. We define $\mathcal{V}_{diag}^\sigma$ for $\sigma \in \mathcal{S}_n$ by

$$\mathcal{V}_{diag}^\sigma = \{X \in \mathcal{A}_\alpha | ||\mathrm{diag}(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)}) - X|| < \epsilon\}.$$

Consider the characteristic polynomial $|\lambda I_n - X| = \lambda^n + K_{n-1}(X)\lambda^{n-1} + \cdots + K_0(X)$. Since $X \in \mathcal{A}_\alpha$, the coefficients of the characteristic polynomial are constants, say, $M_0, \ldots, M_{n-1}$. By solving $K_i(X) = M_i$, $i = 0, \ldots, n-1$, the defining equations become $x_{i,i} = \alpha_{\sigma(i)} + f_i^\sigma(X')$ $i = 1, \ldots, n$ on $\mathcal{V}_{diag}^\sigma$, where X' is $X - \mathrm{diag}(x_{1,1}, \ldots, x_{n,n})$ and $f_i^\sigma(X')$ are algebraic functions of components of X' which satisfies $f_i^\sigma(0) = 0$. We see that $\{\mathcal{V}_i, \mathcal{V}_{diag}^\sigma\}_{i=1,\ldots,n,\sigma\in\mathcal{S}_n}$ are $n^2 - n$ dimensional local coordinate systems. We see that $X\Xi_i(X) = \Xi_i(X)\mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ on $\mathcal{V}_i$, where $\Xi_i(X) = {}^t(P_{i,j}(X, \alpha_k))$. Since $X \in \mathcal{V}_i$, we see that $\Xi_i(X) \in G$. Then we have $X = \Xi_i(X)\mathrm{diag}(\alpha_1, \ldots, \alpha_n)\Xi_i(X)^{-1}$. Put $e^X = (\varphi_{i,j}(X))$.

Proposition 1. $\varphi_{i,j}(X)$ *are rational functions on* $\mathcal{V}_i$.

Proof. We see that $e^X = \Xi_i(X)\mathrm{diag}(e^{\alpha_1}, \ldots, e^{\alpha_n})\Xi_i(X)^{-1}$. Since $\Xi_i(X) = {}^t(P_{i,j}(X, \alpha_k))$, the components of $\Xi_i(X), \Xi_i(X)^{-1}$ and then the components of e^X are rational functions on $\mathcal{V}_i$. $\square$

Proposition 2. $\varphi_{i,j}(X)$ *are algebraic functions on* $\mathcal{V}_{diag}^\sigma$ *for any* $\sigma \in \mathcal{S}_n$.

Proof. We consider the case of $\sigma = id$ and abbreviate $\mathcal{V}_{diag}^{id}$ to $\mathcal{V}$. Put $\xi(\alpha_i) = {}^t(P_{i,1}(X, \alpha_i), \ldots, P_{i,i}(X, \alpha_i), \ldots, P_{i,n}(X, \alpha_i))$ on $\mathcal{V}$. We have $X\xi(\alpha_i) = \alpha_i\xi(\alpha_i)$. Let $\mathcal{I}$ be an ideal of $\mathcal{A}_\alpha$ generated by $x_{\mu,\nu}, \mu \neq \nu$. Then we have

$$(2) \quad P_{i,i}(X, \alpha_i) \equiv (\alpha_i - x_{1,1}) \cdots (\alpha_i - x_{i-1,i-1})$$

$$\times (\alpha_i - x_{i+1,i+1}) \cdots (\alpha_i - x_{n,n}) \mathrm{mod}\mathcal{I}.$$

We see that $x_{j,j} = \alpha_j + f_j(X')$, $j = 1, \ldots, n$ on $\mathcal{V}$. Substituting them, we obtain $\xi(\alpha_i) = m(\alpha_i)\mathbf{e}_i + \mathbf{f}_i(X')$, where $\mathbf{e}_i$ is the i-th elemental

vector, $m(\alpha_i) = \Pi_{j\neq i}(\alpha_i - \alpha_j)$ and $\mathbf{f}_i(X') \in \mathcal{I} \otimes \mathbf{C}^n$. Dividing by $m(\alpha_i)$, we normalize $\xi(\alpha_i)$. Put $\Xi(X) = (\xi(\alpha_1), \ldots, \xi(\alpha_n))$. Then we have $\Xi(X) = E_n + \Psi(X')$, where $\Psi(X') \in \mathcal{I} \otimes Mat(n, \mathbf{C})$. Since $\Psi(0) = 0$, we see that $\Xi(X) \in G$ on $\mathcal{V}$. Then we have $e^X = \Xi(X)\mathrm{diag}(e^{\alpha_1}, \ldots, e^{\alpha_n})\Xi(X)^{-1}$. Since the components of $\Xi(X)$ are algebraic functions of X', we have conclusion. $\square$

3. Compactification of the level sets and Bruhat decomposition. Let us consider the Gauss decomposition for $X \in \mathcal{A}_\alpha$

$$(3) \qquad W_\infty(X)^{-1}W_0(X) = e^X,$$

where $W_\infty(X) \in \bar{N}$ and $W_0(X) \in B$. Put $W_\infty(X) = (w_{i,j}(X))$. We can solve (3) formaly $w_{i,j}(X) = -\tau_{i,j}^\phi(X)/\tau_i^\phi(X)$, where $\tau_i^\phi(X) = \det(\varphi_{k,\ell}(X))_{1\leq k,\ell\leq i-1}$ and $\tau_{i,j}^\phi(X)$ is determinant of matrix whose j-th row is $(\varphi_{i,1}(X), \ldots, \varphi_{i,i-1}(X))$ and other rows are the same as those of $\tau_i^\phi(X)$. We define the divisor Θ_ϕ on $\mathcal{A}_\alpha$ by $\Theta_\phi := \{X \in \mathcal{A}_\alpha | \tau_2^\phi(X) \cdots \tau_n^\phi(X) = 0\}$. Let us consider the algebraic morphism $F_\phi : \mathcal{A}_\alpha - \Theta_\phi \to \bar{N}$ by $F_\phi(X) = W_\infty(X)$. For $\sigma \in \mathcal{S}_n$, we consider the Gauss decomposition

$$(3)_\sigma \qquad W_\infty^\sigma(X)^{-1}W_0^\sigma(X) = \sigma^{-1}e^X,$$

where $W_\infty^\sigma(X) \in \bar{N}$ and $W_0^\sigma(X) \in B$ and we identify σ with $\Sigma_{i=1}^n E_{\sigma(i),i}$. Put $W_\infty^\sigma(X) = (w_{i,j}^\sigma(X))$. We can also solve $(3)_\sigma$ formaly such as $w_{i,j}^\sigma(X) = -\tau_{i,j}^\sigma(X)/\tau_i^\sigma(X)$, where $\tau_i^\sigma(X) = \det(\varphi_{\sigma(k),\ell}(X))_{1\leq k,\ell\leq i-1}$ and $\tau_{i,j}^\sigma(X)$ is the determinant of matrix whose j-th row is $(\varphi_{\sigma(i),1}(X), \ldots, \varphi_{\sigma(i),i-1}(X))$ and other rows are the same as those of $\tau_i^\sigma(X)$. The divisor Θ_σ is defined by $\Theta_\sigma := \{X \in \mathcal{A}_\alpha | \tau_2^\sigma(X) \cdots \tau_n^\sigma(X) = 0\}$. We define the algebraic morphism $F_\sigma : \mathcal{A}_\alpha - \Theta_\sigma \to \bar{N}$ by $F_\sigma(X) = W_\infty^\sigma(X)$. Put $U_\sigma = \mathrm{Im}F_\sigma$ for $\sigma \in \mathcal{S}_n$ ($U_{id} = \mathrm{Im}F_\phi$). Then we obtain $n!$ patches $\{U_\sigma\}_{\sigma\in\mathcal{S}_n}$. We glue together these patches as follows. For $W \in U_\sigma$ and $W' \in U_\tau$, if there exists $X \in \mathcal{A}_\alpha - (\Theta_\sigma \cup \Theta_\tau)$ such that $W = F_\sigma(X)$ and $W' = F_\tau(X)$, then we identify W and W'. This gluing is well defined. Suppose that there exists another $X' \in \mathcal{A}_\alpha - (\Theta_\sigma \cup \Theta_\tau)$ such that $W = F_\sigma(X')$. Then we have $e^{X'} = e^Xb$, where $b = W_0^\sigma(X')^{-1}W_0^\sigma(X) \in B$. We see that

$$F_\tau(X')^{-1}W_0^\tau(X') = \tau^{-1}e^{X'} = \tau^{-1}e^Xb$$
$$= W'^{-1}(W_0^\tau(X)b).$$

By the uniqueness of the Gauss decomposition, we have $F_\tau(X') = W'$.

We denote the resulting variety by $\bar{S}_\alpha$ and call it as the compactification of S_α, where $S_\alpha \subset \mathcal{A}_\alpha$ is the iso level set of the Hessenberg matrices (cf. torus embedding [5]). We show the compactness of $\bar{S}_\alpha$. Put $Z_\sigma = F_\sigma(\Theta_\phi \cap (\mathcal{A}_\alpha - \Theta_\sigma))$ for $\sigma \in \mathcal{S}_n$. Note that the Bruhat decomposition $G/B = \sqcup_{\sigma \in \mathcal{S}_n} \bar{N}\sigma B/B$ induces the cell decomposition of the grand cell such as $\bar{N}B/B = \sqcup_{\sigma \in \mathcal{S}_n} (\bar{N} \cap \sigma^{-1}\bar{N}\sigma)B/B$. Suppose $X \in \Theta_\phi$. There exists unique $\sigma \in \mathcal{S}_n$ satisfying

$$(4) \qquad W_\infty(X)^{-1}W_0(X) = \sigma^{-1}e^X,$$

where $W_\infty(X) \in \bar{N} \cap \sigma^{-1}\bar{N}\sigma$ and $W_0(X) \in B$ from the decomposition of the grand cell mentioned above[4]. Let us explain (4) briefly. $X \in \Theta_\phi$ means e^X does not belong to the grand cell $\bar{N}B/B$. Thus there exists $\sigma \in \mathcal{S}_n$ such that e^X belongs to σ-cell of G/B. Then there exists $\tilde{W}_\infty(X) \in \bar{N}$ and $\tilde{W}_0(X) \in B$ satisfying $\tilde{W}_\infty(X)^{-1}\sigma\tilde{W}_0(X) = e^X$. There exist $n_1, \ldots, n_r \in \bar{N}$ such that $\sigma^{-1}\tilde{W}_\infty(X)^{-1}n_1 \cdots n_r\sigma \in \bar{N}$ and $\sigma^{-1}n_i\sigma \in N, i = 1, \ldots, r$(the proof of the existence of such n_i's is given in [4] which is essentially Gauss' elimination method). Put $W_\infty(X) = \sigma^{-1}n_r^{-1} \cdots n_1^{-1}\tilde{W}_\infty(X)\sigma$. We see that $W_\infty(X) \in \bar{N}$ and $\sigma W_\infty(X)\sigma^{-1} = n_r^{-1} \cdots n_1^{-1}\tilde{W}_\infty(X) \in \bar{N}$. Then we have $W_\infty(X) \in \bar{N} \cap \sigma^{-1}\bar{N}\sigma$. From $\tilde{W}_\infty(X)^{-1}\sigma\tilde{W}_0(X) = e^X$, we have

$$W_\infty(X)^{-1}\sigma^{-1}n_r^{-1} \cdots n_1^{-1}\sigma\tilde{W}_0(X) = \sigma^{-1}e^X.$$

Put $W_0(X) = \sigma^{-1}n_r^{-1} \cdots n_1^{-1}\sigma\tilde{W}_0(X)$. Since

$$\sigma^{-1}n_r^{-1} \cdots n_1^{-1}\sigma = \sigma^{-1}n_r^{-1}\sigma \cdots \sigma^{-1}n_1^{-1}\sigma \in N,$$

we have $W_0(X) \in B$. Then we have (4).

Each point of Θ_ϕ determines point of $\bar{S}_\alpha$ whose representative element is $W_\infty(X) \in \bar{N} \cap \sigma^{-1}\bar{N}\sigma$. Comparing (4) with $(3)_\sigma$, each of representatives belongs to Z_σ. Since each of representative elements does not coincide with any other representative elements under the rule of glueing patches $\{U_\sigma\}_{\sigma \in \mathcal{S}_n}$, we obtain the cell decomposition of the following form

$$(5) \qquad \bar{S}_\alpha = \sqcup_{\sigma \in \mathcal{S}_n} T_\sigma,$$

where the poins of T_σ are consisted of points whose representatives belong to $\bar{N} \cap \sigma^{-1}\bar{N}\sigma$. We see that $T_\sigma \simeq \mathbf{C}^{n(n-1)/2-\ell(\sigma)} \simeq D^{n(n-1)-2\ell(\sigma)}$, where $\ell(\sigma)$ is a number of inversions of σ and D^k is k-dimensional

real open ball, and $\bar{T}_\sigma$ has the cell decomposition $\bar{T}_\sigma = \sqcup_{\tau \geq \sigma} T_\tau$. Then, from the cell decomposition (5), we see that $\bar{S}_\alpha$ is compact and $H^*(\bar{S}_\alpha; \mathbf{Z}) \cong H^*(G/B; \mathbf{Z})$ as $\mathbf{Z}$-module.

4. Cohomology rings of the compactified iso level sets.

For $L \in S_\alpha$, it is known that there exists unique $a \in \bar{N}$ such that $a^{-1}La = \Lambda + \sum_{j=1}^n \ell_j(\alpha)E_{n,j}(= \Lambda(\alpha))$, where $\ell_j(\alpha)$ are functions of α. We consider the trivial bundles $\{U_\sigma \times S_\alpha\}_{\sigma \in \mathcal{S}_n}$. The section on U_σ is defined by $L_\sigma(X) = F_\sigma(X)\Lambda(\alpha)F_\sigma(X)^{-1}$, $X \in \mathcal{A}_\alpha - \Theta_\sigma$. For $X \in \mathcal{A}_\alpha - (\Theta_\sigma \cup \Theta_\tau)$, we have

$$L_\tau(X) = F_\tau(X)\Lambda(\alpha)F_\tau(X)^{-1}$$
$$= \mathrm{Ad}(F_\sigma(X)F_\tau(X)^{-1})^{-1}L_\sigma(X).$$

Then we obtain a principal $\bar{N}$ bundle over $\bar{S}_\alpha$ from $\{U_\sigma \times S_\alpha\}_{\sigma \in \mathcal{S}_n}$. We denote this principal $\bar{N}$ bundle by $p : Lax \to \bar{S}_\alpha$. For the integers $1 \leq i_1 < \cdots < i_k \leq n$, we define the associated fiber bundle $\pi^{i_1, \ldots, i_k} : \mathcal{E}^{i_1, \ldots, i_k} \to \bar{S}_\alpha$ as follows. The fiber on $W \in \bar{S}_\alpha$ is the Grassmann manifold $G_k(\mathbf{C}^n)$

$$\mathcal{E}_W^{i_1, \ldots, i_k} := \{ <\eta_{i_1}, \ldots, \eta_{i_k} >_\mathbf{C} | L\eta_{i_\mu} = \alpha_{i_\mu}\eta_{i_\mu}, \mu$$
$$= 1, \ldots, k, \text{for some } L \in Lax_W \}.$$

Lemma. $\mathcal{E}^{i_1, \ldots, i_k}$ is a principal $\bar{N}$ bundle on $\bar{S}_\alpha$.

Proof. For $a \in \bar{N}$, we define the right action of $\bar{N}$ by $<\eta_{i_1}, \ldots, \eta_{i_k} >_\mathbf{C} a = <a^{-1}\eta_{i_1}, \ldots, a^{-1}\eta_{i_k} >_\mathbf{C}$. Since $L \in Lax_W$ and $L\eta_{i_\mu} = \alpha_{i_\mu}\eta_{i_\mu}$, we have $a^{-1}La \in Lax_W$ and $(a^{-1}La)(a^{-1}\eta_{i_\mu}) = \alpha_{i_\mu}(a^{-1}\eta_{i_\mu})$. Then $<a^{-1}\eta_{i_1}, \ldots, a^{-1}\eta_{i_k} >_\mathbf{C} \in \mathcal{E}_W^{i_1, \ldots, i_k}$. $\square$

In general, let M be a complex manifold. Moreover let $proj : \mathcal{N} \to M$ be a holomorphic fiber bundle and $\tilde{proj} : \mathcal{H} \to \mathcal{N}$ a complex vector bundle. Let $\mathcal{L}^1, \ldots, \mathcal{L}^r$ be complex line bundles over M. We denote the curvature form on $\mathcal{N}$ by $R \in \Omega^2(\mathcal{N}) \otimes End(\mathcal{H})$. Furthermore we denote the first chern class defined by $\mathcal{L}^i$ by $c_1(\mathcal{L}^i) \in H^2(M, \mathbf{Z})$. If $\mathcal{H} \cong \oplus_{i=1}^r proj^*\mathcal{L}^i$ as vector bundles on $\mathcal{N}$, then it holds that

$$\det(\lambda - \frac{1}{2\pi i}R) = \Pi_{i=1}^r (\lambda - proj^*c_1(\mathcal{L}^i)).$$

We define the vector bundle $\varpi^{i_1, \ldots, i_k} : \mathcal{K}^{i_1, \ldots, i_k} \to \mathcal{E}^{i_1, \ldots, i_k}$ as follows. We can write $P \in \mathcal{E}^{i_1, \ldots, i_k}$ as $P = (W, <\eta_{i_1}, \ldots, \eta_{i_k} >_\mathbf{C})$ in local. The fiber on P is a vector space $\oplus_{\mu=1}^k \mathbf{C}\eta_{i_\mu}$. Moreover, the complex line bundles $\mathcal{F}^i$ $i = 1, \ldots, n$ on $\bar{S}_\alpha$ are defined as

follows: The section $L(W)$ of $\Gamma(\bar{S}_\alpha, Lax)$ is defined by $L(W) = W\Lambda(\alpha)W^{-1}$. Let $\xi_i(W) \in \mathbf{C}^n$ be the eigen vector such that $L(W)\xi_i(W) = \alpha_i\xi_i(W)$ and $\|\xi_i(W)\| = 1$. The fiber on $W \in \bar{S}_\alpha$ is defined by $\mathcal{F}_W^i = \mathbf{C}\xi_i(W)$. Let $\mathcal{F}$ be the vector bundle of the direct sum $\oplus_{i=1}^n \mathcal{F}^i$. We abbreviate $\pi^{1,\dots,n} : \mathcal{E}^{1,\dots,n} \to \bar{S}_\alpha$ to $\pi : \mathcal{E} \to \bar{S}_\alpha$ and $\varpi^{1,\dots,n} : \mathcal{K}^{1,\dots,n} \to \mathcal{E}^{1,\dots,n}$ to $\varpi : \mathcal{K} \to \mathcal{E}$. Since $\mathbf{e}_1, \dots, \mathbf{e}_n \in \Gamma(\mathcal{E}, \mathcal{K})$, we see that $\mathcal{K}$ is a trivial bundle, that is, $\mathcal{K} = \mathcal{E} \times \mathbf{C}^n$. Suppose $W \in U_\sigma \cap U_\tau$ and $W = F_\sigma(X)$ on U_σ and $W = F_\tau(X)$ on U_τ for $X \in \mathcal{A}_\alpha - (\Theta_\sigma \cup \Theta_\tau)$. Then we have $\xi_i(F_\tau(X)) = F_\tau(X)F_\sigma(X)^{-1}\xi_i(F_\sigma(X))\psi_{\tau\sigma}^i(W)$, where $\psi_{\tau\sigma}^i(W) \in \mathbf{C}^*$ is a factor to normalize $F_\tau(X)F_\sigma(X)^{-1}\xi_i(F_\sigma(X))$. Transition functions $\psi_{\sigma\tau}^i : U_\sigma \cap U_\tau \to \mathbf{C}^*$ define the 1-st Chern class $c_1(\mathcal{F}^i) \in H^2(\bar{S}_\alpha; \mathbf{Z})$. Let $\pi^* \oplus_{i=1}^n \mathcal{F}^i$ and $\pi^*\mathcal{F}^i$ be the pullback of $\oplus_{i=1}^n \mathcal{F}^i$ and $\mathcal{F}^i$ by π respectively. As vector bundles over $\mathcal{E}$, we see that $\pi^*(\oplus_{i=1}^n \mathcal{F}^i) \cong \oplus_{i=1}^n \pi^*(\mathcal{F}^i)$ and it holds that $\mathcal{K} \cong \oplus_{i=1}^n \pi^*(\mathcal{F}^i)$ as vector bundles on $\mathcal{E}$. From the cell decomposition (5), we see that the cohomology ring of $\bar{S}_\alpha$ is generated by 1-st Chern classes. Since $\mathcal{K}$ is trivial, we see that $H^*(\bar{S}_\alpha; \mathbf{Z}) \cong H^*(G/B; \mathbf{Z})$ as cohomology ring by the following theorem.

Theorem. *It holds that*

$$\sum_{1 \le i_1 < \cdots < i_k \le n} c_1(\mathcal{F}^{i_1}) \cdots c_1(\mathcal{F}^{i_k}) = 0, \ k = 1, \dots, n.$$

Proof. Since $\mathcal{K}$ is trivial, its curvature form R is 0. Since $\mathcal{K} \cong \oplus_{i=1}^n \pi^*(\mathcal{F}^i)$, we have

$$\det(\lambda - \frac{1}{2\pi i}R) = \lambda^n = \Pi_{i=1}^n(\lambda - \pi^*c_1(\mathcal{F}^i)).$$

Then we have

$$\sum_{1 \le i_1 < \cdots < i_k \le n} \pi^*(c_1(\mathcal{F}^{i_1})) \cdots \pi^*(c_1(\mathcal{F}^{i_k}))$$

$$= \pi^*\left(\sum_{1 \le i_1 < \cdots < i_k \le n} c_1(\mathcal{F}^{i_1}) \cdots c_1(\mathcal{F}^{i_k})\right)$$

$= 0$. Since $\pi : \mathcal{E} \to \bar{S}_\alpha$ is surjective, then $\pi^* : H^*(\bar{S}_\alpha; \mathbf{Z}) \to H^*(\mathcal{E}; \mathbf{Z})$ is injective. Hence we have conclusion. $\square$

Application. Take $L_0 \in \bar{S}_\alpha$. Note that $L_0 + \cdots + L_0^n \in \mathcal{A}_{[\alpha]}$, where $[\alpha] = (\alpha_1 + \cdots + \alpha_1^n, \dots, \alpha_n + \cdots + \alpha_n^n)$. Put $\mathbf{t} = (t_1, \dots, t_n) \in (0, 1]^n$. Consider the Gauss decomposition $W_\infty(\mathbf{t})^{-1}W_0(\mathbf{t}) = \exp(t_1 L_0 + \cdots + t_n L_0^n)$. This Gauss decomposition gives orbit of the full Kostant-Toda lattice. Suppose $L_0 + \cdots + L_0^n \in \mathcal{A}_{[\alpha]} - \Theta_\phi$. Then we see that $W_\infty(\mathbf{t})$ belongs to the grand cell near $\mathbf{t} = (1, \dots, 1)$. Note that $\bar{S}_{\mathbf{t}\cdot[\alpha]}$ is homotopic to $\bar{S}_{[\alpha]}$. In finite time, $t_1 L_0 + \cdots + t_n L_0^n$ meets the singular divisor. Then, $W_\infty(\mathbf{t})$ leaves from the large cell. Although $W_\infty(\mathbf{t})$ has pole at this time, the same $W_\infty(\mathbf{t})$ is a usual point in another patch. Then, we obtain the compactification of the orbit of the full Kostant -Toda lattice.

References

[1] N. Ercolani, H. Flaschka and S. Singer, The geometry of the full Kostant-Toda lattice, in *Integrable systems (Luminy*, 1991) 181–225, Prog. Math., 115, Birkhäuser, Boston, 1993.

[2] H. Flaschka and L. Haine, variétés de drapeaux et réseaux de Toda, Math. Z. **208** (1991), 545–556.

[3] Y. Guivarc'h, L. Ji and J. Taylor, *Compactifications of symmetric spaces*, Prog. Math., 156, Birkhäuser Boston, Boston, 1998.

[4] K. Ikeda, The full Kostant-Toda flow associated with the small cell of the flag variety. (Preprint).

[5] B. Shipman, On the geometry of certain isospectral sets in the full Kostant-Toda lattice, Pacific Jour. Math. **181** (1997), 159–185.