Symmetric crystals and affine Hecke algebras of type B

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Abstract: The Lascoux-Leclerc-Thibon conjecture, reformulated and solved by S. Ariki, asserts that the K-group of the representations of the affine Hecke algebras of type A is isomorphic to the algebra of functions on the maximal unipotent subgroup of the group associated with a Lie algebra $\mathfrak{g}$ where $\mathfrak{g}$ is $\mathfrak{gl}_\infty$ or the affine Lie algebra $A_\ell^{(1)}$, and the irreducible representations correspond to the upper global bases. In this note, we formulate analogous conjectures for certain classes of irreducible representations of affine Hecke algebras of type B.

Key words: Crystal bases; affine Hecke algebras; LLT conjecture.

1. Introduction. The purpose of this note is to formulate and explain conjectures on certain classes of irreducible representations of affine Hecke algebras of type B analogous to the Lascoux-Leclerc-Thibon conjecture ([3]), reformulated and solved by S. Ariki, on affine Hecke algebras of type A.

Let us begin by recalling the Lascoux-Leclerc-Thibon conjecture solved by S. Ariki ([1]). Let $\mathrm{H}_n^{\mathrm{A}}$ be the affine Hecke algebra of type A of degree n. Let $\mathrm{K}_n^{\mathrm{A}}$ be the Grothendieck group of the abelian category of finite-dimensional $\mathrm{H}_n^{\mathrm{A}}$-modules, and $\mathrm{K}^{\mathrm{A}} = \oplus_{n\geq 0}\mathrm{K}_n^{\mathrm{A}}$. Then it has a structure of Hopf algebra by the restriction and the induction (cf. §3.3). The set $I = \mathbf{C}^*$ may be regarded as a Dynkin diagram with I as the set of vertices and with edges between $a \in I$ and ap_1^2 (see (7)). Here p_1 is the parameter of the affine Hecke algebra usually denoted by q. Let $\mathfrak{g}_I$ be the associated Lie algebra, and $\mathfrak{g}_I^-$ the unipotent Lie subalgebra. Hence $\mathfrak{g}_I$ is isomorphic to a direct sum of copies of $A_\ell^{(1)}$ if p_1^2 is a primitive ℓ-th root of unity and to a direct sum of copies of $\mathfrak{gl}_\infty$ if p_1 has an infinite order. Let U_I be the group associated to $\mathfrak{g}_I^-$. Then $\mathbf{C}\otimes\mathrm{K}^{\mathrm{A}}$ is isomorphic to the algebra $\mathscr{O}(U_I)$ of regular functions on U_I. Let $U_q(\mathfrak{g}_I)$ be the associated quantized enveloping algebra. Then $U_q^-(\mathfrak{g}_I)$ has an upper global basis $\{G^{\mathrm{up}}(b)\}_{b\in B(\infty)}$. By specializing $\bigoplus \mathbf{C}[q,q^{-1}]G^{\mathrm{up}}(b)$ at $q=1$, we obtain $\mathscr{O}(U_I)$. Then the LLT-conjecture says that the elements associated to irreducible H^{A}-modules corresponds to the image of the upper global basis.

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In this note, we shall formulate analogous conjectures for affine Hecke algebras of type B. In the type B case, we have to replace $U_q^-(\mathfrak{g}_I)$ and its upper global basis with a new object, the symmetric crystals (see § 2). It is roughly stated as follows. Let $\mathrm{H}_n^{\mathrm{B}}$ be the affine Hecke algebra of type B of degree n. Let $\mathrm{K}_n^{\mathrm{B}}$ be the Grothendieck group of the abelian category of finite-dimensional modules over $\mathrm{H}_n^{\mathrm{B}}$, and $\mathrm{K}^{\mathrm{B}} = \oplus_{n\geq 0}\mathrm{K}_n^{\mathrm{B}}$. Then K^{B} has a structure of a Hopf bimodule over K^{A}. The group U_I has an involution θ induced by the involution $a \mapsto a^{-1}$ of $I = \mathbf{C}^*$. Let U_I^θ be the θ-fixed point set of U_I. Then $\mathscr{O}(U_I^\theta)$ is a quotient ring of $\mathscr{O}(U_I)$. The action of $\mathscr{O}(U_I) \simeq \mathbf{C}\otimes\mathrm{K}^{\mathrm{A}}$ on $\mathbf{C}\otimes\mathrm{K}^{\mathrm{B}}$, in fact, descends to the action of $\mathscr{O}(U_I^\theta)$.

We introduce $V_\theta(\lambda)$ (see § 2), a kind of the q-analogue of $\mathscr{O}(U_I^\theta)$. Our conjecture is then:

(i) $V_\theta(\lambda)$ has a crystal basis and a global basis.

(ii) K^{B} is isomorphic to a specialization of $V_\theta(\lambda)$ at $q=1$ as an $\mathscr{O}(U_I)$-module, and the irreducible representations correspond to the upper global basis of $V_\theta(\lambda)$ at $q=1$.

We exclude the representations of $\mathrm{H}_n^{\mathrm{B}}$ such that X_i have an eigenvalue ± 1 (see § 3).

2. Symmetric crystals. In this section, we shall introduce crystals associated with quantum groups with an involution.

2.1. Quantized universal enveloping algebras. We shall recall the quantized universal enveloping algebra $U_q(\mathfrak{g})$. Let I be an index set (for simple roots), and Q the free $\mathbf{Z}$-module with a basis $\{\alpha_i\}_{i\in I}$. Let $(\,\bullet\,,\,\bullet\,)\colon Q\times Q\to\mathbf{Z}$ be a symmetric bilinear form such that $(\alpha_i, \alpha_i)/2 \in \mathbf{Z}_{>0}$ for any i and $(\alpha_i^\vee, \alpha_j) \in \mathbf{Z}_{\le 0}$ for $i \neq j$ where $\alpha_i^\vee := 2\alpha_i/(\alpha_i, \alpha_i)$. Let q be an indeterminate and set $K := \mathbf{Q}(q)$. We define its subrings $\mathbf{A}_0$, $\mathbf{A}_\infty$ and $\mathbf{A}$ as follows:

$$\begin{aligned}\mathbf{A}_0 &= \{f/g\,;\, f(q), g(q) \in \mathbf{Q}[q],\, g(0) \neq 0\},\\ \mathbf{A}_\infty &= \{f/g\,;\, f(q^{-1}), g(q^{-1}) \in \mathbf{Q}[q^{-1}],\\ &\qquad\qquad g(q^{-1})|_{q^{-1}=0} \neq 0\},\\ \mathbf{A} &= \mathbf{Q}[q, q^{-1}].\end{aligned}$$

Definition 2.1. *The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is the K-algebra generated by the elements e_i, f_i and invertible elements t_i $(i \in I)$ with the following defining relations.*

(i) *The t_i's commute with each other.*

(ii) $t_j e_i t_j^{-1} = q^{(\alpha_j, \alpha_i)} e_i$ *and* $t_j f_i t_j^{-1} = q^{-(\alpha_j,\alpha_i)} f_i$ *for any $i, j \in I$.*

(iii) $[e_i, f_j] = \delta_{ij} \dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$ *for i, $j \in I$. Here* $q_i := q^{(\alpha_i,\alpha_i)/2}$.

(iv) (Serre relation) *For $i \neq j$,*

$$\sum_{k=0}^{b} (-1)^k e_i^{(k)} e_j e_i^{(b-k)} = 0,$$
$$\sum_{k=0}^{b} (-1)^k f_i^{(k)} f_j f_i^{(b-k)} = 0.$$

Here $b = 1 - (\alpha_i^\vee, \alpha_j)$ and

$$\begin{aligned} e_i^{(k)} &= e_i^k/[k]_i!\,,\ f_i^{(k)} = f_i^k/[k]_i!\,,\\ [k]_i &= (q_i^k - q_i^{-k})/(q_i - q_i^{-1})\,,\\ [k]_i! &= [1]_i \cdots [k]_i\,. \end{aligned}$$

Let us denote by $U_q^-(\mathfrak{g})$ (resp. $U_q^+(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by the f_i's (resp. the e_i's). Let us recall the crystal theory of $U_q^-(\mathfrak{g})$ ([4]). Let e_i' and e_i^* be the operators on $U_q^-(\mathfrak{g})$ defined by

$$[e_i, a] = \frac{(e_i^* a) t_i - t_i^{-1} e_i' a}{q_i - q_i^{-1}} \quad (a \in U_q^-(\mathfrak{g})).$$

Then these operators satisfy the following formula similar to derivations:

$$\begin{aligned} e_i'(ab) &= e_i'(a)b + (\mathrm{Ad}(t_i)a)e_i'b,\\ e_i^*(ab) &= a e_i^* b + (e_i^* a)(\mathrm{Ad}(t_i)b). \end{aligned}$$

Then $U_q^-(\mathfrak{g})$ has a unique symmetric bilinear form $(\bullet, \bullet)$ such that $(1,1) = 1$ and

$$(e_i' a, b) = (a, f_i b) \quad \text{for any } a, b \in U_q^-(\mathfrak{g}).$$

It is non-degenerate and satisfies $(e_i^* a, b) = (a, b f_i)$. The left multiplication of f_j and e_i' have the commutation relation

$$e_i' f_j = q^{-(\alpha_i, \alpha_j)} f_j e_i' + \delta_{ij},$$

and both the e_i''s and the f_i's satisfy the Serre relations. Since e_i' and f_i satisfy the q-boson relation, any element $a \in U_q^-(\mathfrak{g})$ can be written uniquely as

$$a = \sum_{n \ge 0} f_i^{(n)} a_n \quad \text{with } e_i' a_n = 0.$$

We define the modified root operators by

$$(1)\qquad \begin{aligned} \tilde{e}_i a &= \textstyle\sum_{n\ge 1} f_i^{(n-1)} a_n \quad \text{and}\\ \tilde{f}_i a &= \textstyle\sum_{n \ge 0} f_i^{(n+1)} a_n. \end{aligned}$$

Let $L(\infty)$ be the $\mathbf{A}_0$-submodule of $U_q^-(\mathfrak{g})$ generated by the $\tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} 1$'s ($\ell \ge 0$, $i_1, \ldots, i_\ell \in I$). Let $B(\infty)$ be the subset $\left\{\tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} 1 \bmod qL(\infty)\,;\, \ell \ge 0,\, i_1, \ldots, i_\ell \in I\right\}$ of $L(\infty)/qL(\infty)$. Then we have

Theorem 2.2. (i) $\tilde{f}_i L(\infty) \subset L(\infty)$ *and* $\tilde{e}_i L(\infty) \subset L(\infty)$,

(ii) *$B(\infty)$ is a basis of $L(\infty)/qL(\infty)$,*

(iii) $\tilde{f}_i B(\infty) \subset B(\infty)$ *and* $\tilde{e}_i B(\infty) \subset B(\infty) \sqcup \{0\}$.

2.2. Global bases. Let $^-$ be the automorphism of K sending q to q^{-1}. Then $\overline{\mathbf{A}_0}$ coincides with $\mathbf{A}_\infty$. Let V be a vector space over K, L_0 an A-submodule of V, L_∞ an $\mathbf{A}_\infty$- submodule, and $V_\mathbf{A}$ an $\mathbf{A}$-submodule. Set $E := L_0 \cap L_\infty \cap V_\mathbf{A}$.

Definition 2.3 ([4])**.** *We say that $(L_0, L_\infty, V_\mathbf{A})$ is* balanced *if each of L_0, L_∞ and $V_\mathbf{A}$ generates V as a K-vector space, and if one of the following equivalent conditions is satisfied.*

(i) *$E \to L_0/qL_0$ is an isomorphism,*

(ii) *$E \to L_\infty/q^{-1}L_\infty$ is an isomorphism,*

(iii) *$(L_0 \cap V_\mathbf{A}) \oplus (q^{-1}L_\infty \cap V_\mathbf{A}) \to V_\mathbf{A}$ is an isomorphism.*

(iv) *$\mathbf{A}_0 \otimes_\mathbf{Q} E \to L_0$, $\mathbf{A}_\infty \otimes_\mathbf{Q} E \to L_\infty$, $\mathbf{A} \otimes_\mathbf{Q} E \to V_\mathbf{A}$ and $K \otimes_\mathbf{Q} E \to V$ are isomorphisms.*

Let $^-$ be the ring automorphism of $U_q(\mathfrak{g})$ sending q, t_i, e_i, f_i to q^{-1}, t_i^{-1}, e_i, f_i.

Let $U_q(\mathfrak{g})_\mathbf{A}$ be the $\mathbf{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}$, $f_i^{(n)}$ and t_i. Similarly we define $U_q^-(\mathfrak{g})_\mathbf{A}$.

Theorem 2.4. $(L(\infty), L(\infty)^-, U_q^-(\mathfrak{g})_\mathbf{A})$ *is balanced.*

Let

$$G\colon L(\infty)/qL(\infty) \xrightarrow{\sim} E := L(\infty) \cap L(\infty)^- \cap U_q^-(\mathfrak{g})_{\mathbf{A}}$$

be the inverse of $E \xrightarrow{\sim} L(\infty)/qL(\infty)$. Then $\{G(b)\,;\, b \in B(\infty)\}$ forms a basis of $U_q^-(\mathfrak{g})$. We call it a (lower) *global basis*. It is first introduced by G. Lusztig ([5]) under the name of "canonical basis" for the A,D,E cases.

2.3. Symmetry. Let θ be an automorphism of I such that $\theta^2 = \mathrm{id}$ and $(\alpha_{\theta(i)}, \alpha_{\theta(j)}) = (\alpha_i, \alpha_j)$. Hence it extends to an automorphism of the root lattice Q by $\theta(\alpha_i) = \alpha_{\theta(i)}$, and induces an automorphism of $U_q(\mathfrak{g})$.

Let $\mathcal{B}_\theta(\mathfrak{g})$ be the K-algebra generated by E_i, F_i, and invertible elements T_i $(i \in I)$ satisfying the followin gdefining relations:

(i) the T_i's commute with each other,

(ii) $T_{\theta(i)} = T_i$ for any i,

(iii) $T_i E_j T_i^{-1} = q^{(\alpha_i + \alpha_{\theta(i)}, \alpha_j)} E_j$ and $T_i F_j T_i^{-1} = q^{(\alpha_i + \alpha_{\theta(i)}, -\alpha_j)} F_j$ for $i, j \in I$,

(iv) $E_i F_j = q^{-(\alpha_i, \alpha_j)} F_j E_i + (\delta_{i,j} + \delta_{\theta(i),j} T_i)$ for i, $j \in I$,

(v) the E_i's and the F_i's satisfy the Serre relations.

Hence $\mathcal{B}_\theta(\mathfrak{g}) \simeq U_q^-(\mathfrak{g}) \otimes K[T_i^{\pm 1}; i \in I] \otimes U_q^+(\mathfrak{g})$. We set $E_i^{(n)} = E_i^n/[n]_i!$ and $F_i^{(n)} = F_i^n/[n]_i!$.

Let $\lambda \in P_+ := \{\lambda \in \mathrm{Hom}(Q, \mathbf{Q})\,;\, \langle \alpha_i^\vee, \lambda\rangle \in \mathbf{Z}_{\geq 0}$ for any $i \in I\}$ be a dominant integral weight such that $\theta(\lambda) = \lambda$.

Proposition 2.5. (i) *There exists a $\mathcal{B}_\theta(\mathfrak{g})$-module $V_\theta(\lambda)$ generated by a vector ϕ_λ such that*

(a) $E_i\phi_\lambda = 0$ *for any* $i \in I$,

(b) $T_i\phi_\lambda = q^{(\alpha_i, \lambda)}\phi_\lambda$ *for any* $i \in I$,

(c) $\{u \in V_\theta(\lambda)\,;\, E_i u = 0$ *for any* $i \in I\} = K\phi_\lambda$.

Moreover such a $V_\theta(\lambda)$ is irreducible and unique up to an isomorphism.

(ii) *there exists a unique symmetric bilinear form $(\bullet, \bullet)$ on $V_\theta(\lambda)$ such that $(\phi_\lambda, \phi_\lambda) = 1$ and $(E_i u, v) = (u, F_i v)$ for any $i \in I$ and $u, v \in V_\theta(\lambda)$, and it is non-degenerate.*

The pair $(\mathcal{B}_\theta(\mathfrak{g}), V_\theta(\lambda))$ is an analogue of $(\mathcal{B}, U_q^-(\mathfrak{g}))$. Such a $V_\theta(\lambda)$ is constructed as follows. Let $U_q^-(\mathfrak{g})\phi'_\lambda$ and $U_q^-(\mathfrak{g})\phi''_\lambda$ be a copy of a free $U_q^-(\mathfrak{g})$-module. We give the structure of a $\mathcal{B}_\theta(\mathfrak{g})$-module on them as follows: for any $i \in I$ and $a \in U_q^-(\mathfrak{g})$

$$(2)\quad \begin{cases} T_i(a\phi'_\lambda) = q^{(\alpha_i,\lambda)}(\mathrm{Ad}(t_i t_{\theta(i)})a)\phi'_\lambda, \\ E_i(a\phi'_\lambda) = \bigl(e'_i a + q^{(\alpha_i,\lambda)}\,\mathrm{Ad}(t_i)(e^*_{\theta(i)}a)\bigr)\phi'_\lambda, \\ F_i(a\phi'_\lambda) = (f_i a)\phi'_\lambda \end{cases}$$

and

$$(3)\quad \begin{cases} T_i(a\phi''_\lambda) = q^{(\alpha_i,\lambda)}(\mathrm{Ad}(t_i t_{\theta(i)})a)\phi''_\lambda, \\ E_i(a\phi''_\lambda) = (e'_i a)\phi''_\lambda, \\ F_i(a\phi''_\lambda) = \bigl(f_i a + q^{(\alpha_i,\lambda)}(\mathrm{Ad}(t_i)a)f_{\theta(i)}\bigr)\phi''_\lambda. \end{cases}$$

Then there exists a unique $\mathcal{B}_\theta(\mathfrak{g})$-linear morphism $\psi\colon U_q^-(\mathfrak{g})\phi'_\lambda \to U_q^-(\mathfrak{g})\phi''_\lambda$ sending ϕ'_λ to ϕ''_λ. Its image $\psi(U_q^-(\mathfrak{g})\phi'_\lambda)$ is $V_\theta(\lambda)$.

Hereafter we assume further that

$$(4)\qquad \text{there is no } i \in I \text{ such that } \theta(i) = i.$$

We conjecture that $V_\theta(\lambda)$ has a crystal basis. This means the following. We define the modified root operators similarly to (1):

$$\tilde{E}_i(u) = \sum_{n\geq 1} F_i^{(n-1)} u_n \text{ and } \tilde{F}_i(u) = \sum_{n \geq 0} F_i^{(n+1)} u_n$$

when writing $u = \sum_{n\geq 0} F_i^{(n)} u_n$ with $E_i u_n = 0$. Let $L_\theta(\lambda)$ be the $\mathbf{A}_0$-submodule of $V_\theta(\lambda)$ generated by $\tilde{F}_{i_1}\cdots\tilde{F}_{i_\ell}\phi_\lambda$ ($\ell \geq 0$ and $i_1, \dots, i_\ell \in I$), and let $B_\theta(\lambda)$ be the subset $\bigl\{\tilde{F}_{i_1}\cdots\tilde{F}_{i_\ell}\phi_\lambda \bmod qL_\theta(\lambda)\,;\, \ell \geq 0,\, i_1, \dots, i_\ell \in I\bigr\}$ of $L_\theta(\lambda)/qL_\theta(\lambda)$.

Conjecture 2.6. (i) $\tilde{F}_i L_\theta(\lambda) \subset L_\theta(\lambda)$ *and* $\tilde{E}_i L_\theta(\lambda) \subset L_\theta(\lambda)$,

(ii) $B_\theta(\lambda)$ *is a basis of* $L_\theta(\lambda)/qL_\theta(\lambda)$,

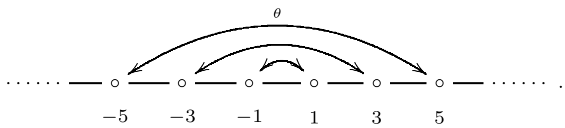
(iii) $\tilde{F}_i B_\theta(\lambda) \subset B_\theta(\lambda)$, *and* $\tilde{E}_i B_\theta(\lambda) \subset B_\theta(\lambda) \sqcup \{0\}$.

Moreover we conjecture that $V_\theta(\lambda)$ has a global crystal basis. Namely, let $-$ be the bar-operator of $V_\theta(\lambda)$ given by $-\colon a\phi_\lambda \to \bar{a}\phi_\lambda$ ($a \in U_q^-(\mathfrak{g})$) (such an operator exists).

Conjecture 2.7. $(L_\theta(\lambda), L_\theta(\lambda)^-, U_q^-(\mathfrak{g})_{\mathbf{A}}\phi_\lambda)$ *is balanced.*

Assume that this conjecture is true. Let $G^{\mathrm{low}}\colon L_\theta(\lambda)/qL_\theta(\lambda) \xrightarrow{\sim} E := L_\theta(\lambda) \cap L_\theta(\lambda)^- \cap U_q^-(\mathfrak{g})_{\mathbf{A}}\phi_\lambda$ be the inverse of $E \xrightarrow{\sim} L_\theta(\lambda)/qL_\theta(\lambda)$. Then $\bigl\{G^{\mathrm{low}}(b)\,;\, b \in B_\theta(\lambda)\bigr\}$ forms a basis of $V_\theta(\lambda)$. We call this basis the *lower global basis* of $V_\theta(\lambda)$. Let $\{G^{\mathrm{up}}(b)\,;\, b \in B_\theta(\lambda)\}$ be the dual basis to $\bigl\{G^{\mathrm{low}}(b)\,;\, b \in B_\theta(\lambda)\bigr\}$ with respect to the inner product of $V_\theta(\lambda)$. We call it the *upper global basis* of $V_\theta(\lambda)$.

We can prove the conjectures in the $\mathfrak{gl}_\infty$-case:



Theorem 2.8. *Let I be the set $\mathbf{Z}_{\mathrm{odd}}$ of odd integers. Define*

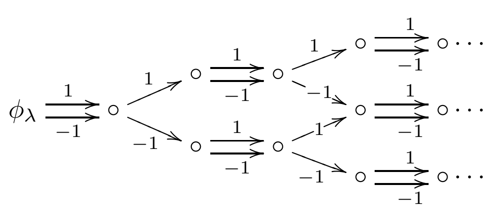
$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 2, \\ 0 & \text{otherwise,} \end{cases}$$

and $\theta(i) = -i$. Then, for $\lambda = 0$, $V_\theta(\lambda)$ has a crystal basis and a global basis.

Note that $\{a \in U_q^-(\mathfrak{g})\,;\, a\phi_\lambda = 0\} = \sum_i U_q^-(\mathfrak{g})(f_i - f_{\theta(i)})$ in this case.

The proof is by using a kind of PBW basis, similarly to [5]. The details will appear elsewhere.

The following diagram is the part of the crystal graph of $B_\theta(\lambda)$ that concerns only the 1-arrows and the (-1)-arrows.



Here is the part of the crystal graph of $B_\theta(\lambda)$ that concerns only the n-arrows and the $(-n)$-arrows for an odd integer $n \geq 3$:

$$\phi_\lambda \overset{n}{\underset{-n}{\rightrightarrows}} \circ \overset{n}{\underset{-n}{\rightrightarrows}} \circ \overset{n}{\underset{-n}{\rightrightarrows}} \circ \overset{n}{\underset{-n}{\rightrightarrows}} \circ \cdots$$

3. Affine Hecke algebra of type B.

3.1. Definition. For $p_0, p_1 \in \mathbf{C}^*$ and $n \in \mathbf{Z}_{\geq 0}$, the affine Hecke algebra $\mathrm{H}_n^{\mathrm{B}}$ of type B_n is the $\mathbf{C}$-algebra generated by T_i $(0 \leq i < n)$ and invertible elements X_i $(1 \leq i \leq n)$ satisfying the defining relations:

(i) the X_i's commute with each other,

(ii) the T_i's satisfy the braid relation: $T_0T_1T_0T_1 = T_1T_0T_1T_0$, $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ $(1 \leq i < n-1)$, $T_iT_j = T_jT_i$ $(|i-j| > 1)$,

(iii) $(T_0 - p_0)(T_0 + p_0^{-1}) = 0$ and $(T_i - p_1)(T_i + p_1^{-1}) = 0$ $(1 \leq i < n)$,

(iv) $T_0X_1^{-1}T_0 = X_1$, $T_iX_iT_i = X_{i+1}$ $(1 \leq i < n)$, and $T_iX_j = X_jT_i$ if $j \neq i, i+1$.

We assume that $p_0, p_1 \in \mathbf{C}^*$ satisfy

$$p_0^2 \neq 1, p_1^2 \neq 1. \tag{5}$$

Let us denote by $\mathbf{Pol}_n$ the Laurent polynomial ring $\mathbf{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, and by $\widetilde{\mathbf{Pol}}_n$ its quotient field $\mathbf{C}(X_1, \ldots, X_n)$. Then $\mathrm{H}_n^{\mathrm{B}}$ is isomorphic to the tensor product of $\mathbf{Pol}_n$ and the subalgebra generated by the T_i's that is isomorphic to the Hecke algebra of type B_n. We have

$$T_i a = (s_i a)T_i + (p_i - p_i^{-1})\frac{a - s_i a}{1 - X^{-\alpha_i^\vee}} \quad \text{for } a \in \mathbf{Pol}_n.$$

Here $p_i = p_1$ $(1 < i < n)$, and $X^{-\alpha_i^\vee} = X_1^{-2}$ $(i = 0)$ and $X^{-\alpha_i^\vee} = X_iX_{i+1}^{-1}$ $(1 \leq i < n)$. The s_i's are the Weyl group action on $\mathbf{Pol}_n$: $(s_i a)(X_1, \ldots, X_n) = a(X_1^{-1}, X_2, \ldots, X_n)$ for $i = 0$ and $(s_i a)(X_1, \ldots, X_n) = a(X_1, \ldots, X_{i+1}, X_i, \ldots, X_n)$ for $1 \leq i < n$.

Note that $\mathrm{H}_n^{\mathrm{B}} = \mathbf{C}$ for $n = 0$.

3.2. Intertwiner. The algebra $\mathrm{H}_n^{\mathrm{B}}$ acts faithfully on $\mathrm{H}_n^{\mathrm{B}} / \sum_i \mathrm{H}_n^{\mathrm{B}}(T_i - p_i) \simeq \mathbf{Pol}_n$. Set $\varphi_i = (1 - X^{-\alpha_i^\vee})T_i - (p_i - p_i^{-1}) \in \mathrm{H}_n^{\mathrm{B}}$ and $\tilde{\varphi}_i = (p_i^{-1} - p_i X^{-\alpha_i^\vee})^{-1}\varphi_i \in \widetilde{\mathbf{Pol}}_n \otimes_{\mathbf{Pol}_n} \mathrm{H}_n^{\mathrm{B}}$. Then the action of $\tilde{\varphi}_i$ on $\mathbf{Pol}_n$ coincides with s_i. They are called intertwiners.

3.3. Affine Hecke algebra of type A. We will review the LLT conjecture, reformulated and solved by S. Ariki, on the affine Hecke algebras of type A.

The affine Hecke algebra $\mathrm{H}_n^{\mathrm{A}}$ of type A_n is isomorphic to the subalgebra of $\mathrm{H}_n^{\mathrm{B}}$ generated by T_i $(1 \leq i < n)$ and $X_i^{\pm 1}$ $(1 \leq i \leq n)$. For a finite-dimensional $\mathrm{H}_n^{\mathrm{A}}$-module M let us decompose

$$M = \bigoplus_{a \in (\mathbf{C}^*)^n} M_a \tag{6}$$

where $M_a = \{u \in M\,;\, (X_i - a_i)^N u = 0 \text{ for any } i \text{ and } N \gg 0\}$ for $a = (a_1, \ldots, a_n) \in (\mathbf{C}^*)^n$. For a subset $I \subset \mathbf{C}^*$, we say that M is of type I if all the eigenvalues of X_i belong to I. The group $\mathbf{Z}$ acts on $\mathbf{C}^*$ by $\mathbf{Z} \ni n\colon a \mapsto ap_1^{2n}$.

Lemma 3.1. *Let I and J be $\mathbf{Z}$-invariant subsets in $\mathbf{C}^*$ such that $I \cap J = \emptyset$.*

(i) *If M is an irreducible $\mathrm{H}_m^{\mathrm{A}}$-module of type I and N is an irreducible $\mathrm{H}_n^{\mathrm{A}}$-module of type J, then $\mathrm{Ind}_{\mathrm{H}_m^{\mathrm{A}} \otimes \mathrm{H}_n^{\mathrm{A}}}^{\mathrm{H}_{m+n}^{\mathrm{A}}}(M \otimes N)$ is irreducible of type $I \cup J$.*

(ii) *Conversely if L is an irreducible $\mathrm{H}_n^{\mathrm{A}}$-module of type $I \cup J$, then there exist m $(0 \leq m \leq n)$, an irreducible $\mathrm{H}_m^{\mathrm{A}}$-module M of type I and an irreducible $\mathrm{H}_{n-m}^{\mathrm{A}}$-module N of type J such that L is isomorphic to $\mathrm{Ind}_{\mathrm{H}_m^{\mathrm{A}} \otimes \mathrm{H}_{n-m}^{\mathrm{A}}}^{\mathrm{H}_n^{\mathrm{A}}}(M \otimes N)$.*

Hence in order to study the irreducible modules over the affine Hecke algebras of type A, it is enough to treat the irreducible modules of type I for an orbit I with respect to the $\mathbf{Z}$-action on $\mathbf{C}^*$. Let $\mathrm{K}^{\mathrm{A}}_{I,n}$ be the Grothendieck group of the abelian category of finite-dimensional $\mathrm{H}^{\mathrm{A}}_n$-modules of type I. We set $\mathrm{K}^{\mathrm{A}}_I = \bigoplus_{n\geq 0}\mathrm{K}^{\mathrm{A}}_{I,n}$. Then $\mathrm{K}^{\mathrm{A}}_I$ has a structure of Hopf algebra where the product and the coproduct

$$\mu\colon\ \mathrm{K}^{\mathrm{A}}_{I,m}\otimes\mathrm{K}^{\mathrm{A}}_{I,n}\to\mathrm{K}^{\mathrm{A}}_{I,m+n},$$
$$\Delta\colon\ \mathrm{K}^{\mathrm{A}}_{I,n}\to\bigoplus_{i+j=n}\mathrm{K}^{\mathrm{A}}_{I,i}\otimes\mathrm{K}^{\mathrm{A}}_{I,j}$$

are given by $M\otimes N\mapsto \mathrm{Ind}^{\mathrm{H}^{\mathrm{A}}_{m+n}}_{\mathrm{H}^{\mathrm{A}}_m\otimes\mathrm{H}^{\mathrm{A}}_n}(M\otimes N)$ and by $M\mapsto \mathrm{Res}^{\mathrm{H}^{\mathrm{A}}_n}_{\mathrm{H}^{\mathrm{A}}_i\otimes\mathrm{H}^{\mathrm{A}}_j}M$. Let $\mathfrak{g}_I$ be the Lie algebra associated to the Dynkin diagram with I as the set of vertices and with edges between a and ap_1^2 $(a\in I)$. It means

$$(7)\quad (\alpha_i,\alpha_j)=2\delta_{i,j}-\delta_{i,p_1^2j}-\delta_{p_1^2i,j}\quad\text{for } i,j\in I.$$

Let U_I be the unipotent group associated with the Lie subalgebra $\mathfrak{g}_I^-$ of $\mathfrak{g}_I$ generated by the f_i's. Then we have

Lemma 3.2. *Let I be a $\mathbf{Z}$-invariant set. Then $\mathbf{C}\otimes\mathrm{K}^{\mathrm{A}}_I$ is isomorphic to the algebra $\mathscr{O}(U_I)$ of the regular functions on U_I as a Hopf algebra.*

Here, for $a\in I$, f_a corresponds to the one-dimensional $\mathrm{H}^{\mathrm{A}}_1$-module $\mathbf{C}_a$ on which X_1 acts by a. Let $\{G^{\mathrm{up}}(b)\}_{b\in B(\infty)}$ be the upper global basis of $U_q^-(\mathfrak{g})$. Then $\big(\bigoplus\mathbf{C}[q,q^{-1}]G^{\mathrm{up}}(b)\big)/\big((q-1)\bigoplus\mathbf{C}[q,q^{-1}]G^{\mathrm{up}}(b)\big)$ is isomorphic to $\mathscr{O}(U_I)$. The following theorem is conjectured for Hecke algebras of type A by Lascoux-Leclerc-Thibon ([3]) and reformulated and proved by S. Ariki ([1]) for affine Hecke algebras of type A.

Theorem 3.3. *The elements of K^{A} associated to irreducible H^{A}-modules correspond to the upper global basis $G^{\mathrm{up}}(b)$ by the isomorphism above.*

Hence the irreducible modules are parametrized by $B(\infty)$. Grojnowski ([2]) constructed the operators $\tilde{e}_a$ and $\tilde{f}_a$ on $B(\infty)$ in terms of irreducible modules. The operator $\tilde{e}_a$ sends an irreducible $\mathrm{H}^{\mathrm{A}}_n$ module M to a unique irreducible submodule of the $\mathrm{H}^{\mathrm{A}}_{n-1}$-module $\{u\in M\,;(X_n-a)u=0\}$. The operator $\tilde{f}_a$ sends an irreducible $\mathrm{H}^{\mathrm{A}}_n$ module M to a unique irreducible quotient of the $\mathrm{H}^{\mathrm{A}}_{n+1}$-module $\mathrm{Ind}^{\mathrm{H}^{\mathrm{A}}_{n+1}}_{\mathrm{H}^{\mathrm{A}}_n\otimes\mathrm{H}^{\mathrm{A}}_1}(M\otimes\mathbf{C}_a)$.

3.4. Representations of affine Hecke algebras of type B. For $n,m\geq 0$, set $\mathbf{F}_{n,m}:=\mathbf{C}[X_1^{\pm1},\dots,X_{n+m}^{\pm1},D^{-1}]$ where $D:=\prod_{1\leq i\leq n<j\leq n+m}(X_i-p_1^2X_j)(X_i-p_1^{-2}X_j)(X_i-p_1^2X_j^{-1})(X_i-p_1^{-2}X_j^{-1})(X_i-X_j)$. Then we can embed $\mathrm{H}^{\mathrm{B}}_m$ into $\mathrm{H}^{\mathrm{B}}_{n+m}\otimes_{\mathbf{P}\mathrm{ol}_{n+m}}\mathbf{F}_{n,m}$ by

$$T_0\mapsto\tilde{\varphi}_n\cdots\tilde{\varphi}_1T_0\tilde{\varphi}_1\cdots\tilde{\varphi}_n,$$
$$T_i\mapsto T_{i+n}\ (1\leq i<m),\ X_i\mapsto X_{i+n}\ (1\leq i\leq m).$$

Its image commute with $\mathrm{H}^{\mathrm{B}}_n\subset\mathrm{H}^{\mathrm{B}}_{n+m}$. Hence $\mathrm{H}^{\mathrm{B}}_{n+m}\otimes_{\mathbf{P}\mathrm{ol}_{n+m}}\mathbf{F}_{n,m}$ is a right $\mathrm{H}^{\mathrm{B}}_n\otimes\mathrm{H}^{\mathrm{B}}_m$-module.

Lemma 3.4. $\mathrm{H}^{\mathrm{A}}_{n+m}\underset{\mathrm{H}^{\mathrm{A}}_n\otimes\mathrm{H}^{\mathrm{A}}_m}{\otimes}\big(\mathrm{H}^{\mathrm{B}}_n\otimes\mathrm{H}^{\mathrm{B}}_m\big)\otimes_{\mathbf{P}\mathrm{ol}_{n+m}}\mathbf{F}_{n,m}\xrightarrow{\sim}\mathrm{H}^{\mathrm{B}}_{n+m}\otimes_{\mathbf{P}\mathrm{ol}_{n+m}}\mathbf{F}_{n,m}$.

For a finite-dimensional $\mathrm{H}^{\mathrm{B}}_n$-module M, we decompose M as in (6). The semidirect product group $\mathbf{Z}_2\times\mathbf{Z}=\{1,-1\}\times\mathbf{Z}$ acts on $\mathbf{C}^*$ by $(\epsilon,n)\colon a\mapsto a^{\epsilon}p_1^{2n}$.

Let I and J be $\mathbf{Z}_2\times\mathbf{Z}$-invariant subsets of $\mathbf{C}^*$ such that $I\cap J=\emptyset$. Then for an $\mathrm{H}^{\mathrm{B}}_n$-module N of type I and $\mathrm{H}^{\mathrm{B}}_m$-module M of type J, the action of $\mathbf{P}\mathrm{ol}_{n+m}$ on $N\otimes M$ extends to an action of $\mathbf{F}_{n,m}$. We set

$$N\diamond M:=(\mathrm{H}^{\mathrm{B}}_{n+m}\otimes_{\mathbf{P}\mathrm{ol}_{n+m}}\mathbf{F}_{n,m})\underset{(\mathrm{H}^{\mathrm{B}}_n\otimes\mathrm{H}^{\mathrm{B}}_m)\otimes_{\mathbf{P}\mathrm{ol}_{n+m}}\mathbf{F}_{n,m}}{\otimes}(N\otimes M).$$

By the lemma above, $N\diamond M$ is isomorphic to $\mathrm{Ind}^{\mathrm{H}^{\mathrm{A}}_{n+m}}_{\mathrm{H}^{\mathrm{A}}_n\otimes\mathrm{H}^{\mathrm{A}}_m}(N\otimes M)$ as an $\mathrm{H}^{\mathrm{A}}_{n+m}$-module.

Lemma 3.5. (i) *Let N be an irreducible $\mathrm{H}^{\mathrm{B}}_n$-module of type I and M an irreducible $\mathrm{H}^{\mathrm{B}}_m$-module of type J. Then $N\diamond M$ is an irreducible $\mathrm{H}^{\mathrm{B}}_{n+m}$-module of type $I\cup J$.*

(ii) *Conversely if L is an irreducible $\mathrm{H}^{\mathrm{B}}_n$-module of type $I\cup J$, then there exist an integer m $(0\leq m\leq n)$, an irreducible $\mathrm{H}^{\mathrm{B}}_m$-module N of type I and an irreducible $\mathrm{H}^{\mathrm{B}}_{n-m}$-module M of type J such that $L\simeq N\diamond M$.*

(iii) *Assume that a $\mathbf{Z}_2\times\mathbf{Z}$-orbit I decomposes into $I=I_+\sqcup I_-$ where $I_\pm$ are $\mathbf{Z}$-orbits and $I_-=(I_+)^{-1}$. Assume that $\pm1,\pm p\notin I$. Then for any irreducible $\mathrm{H}^{\mathrm{B}}_n$-module L of type I, there exists an irreducible $\mathrm{H}^{\mathrm{A}}_n$-module M such that $L\simeq\mathrm{Ind}^{\mathrm{H}^{\mathrm{B}}_n}_{\mathrm{H}^{\mathrm{A}}_n}M$.*

Hence in order to study H^{B}-modules, it is enough to study irreducible modules of type I for a $\mathbf{Z}_2\times\mathbf{Z}$-orbit I in $\mathbf{C}^*$ such that I is a $\mathbf{Z}$-orbit or I contains one of $\pm1,\pm p$.

In this note, *we don't treat the case when I contains* 1 *or* -1.

For a $\mathbf{Z}_2\times\mathbf{Z}$-invariant subset I of $\mathbf{C}^*$, we define $\mathrm{K}^{\mathrm{B}}_{I,n}$ and $\mathrm{K}^{\mathrm{B}}_I$ similarly to the case of A-type. Then

K_I^B is a (right) Hopf K_I^A-bimodule by the multiplication and the coproduct

$$\mu\colon K_{I,n}^B \times K_{I,m}^A \to K_{I,n+m}^B \text{ and}$$
$$\Delta\colon K_{I,n}^B \to \bigoplus_{i+j=n} K_{I,i}^B \otimes K_{I,j}^A$$

given by $L \otimes M \mapsto \mathrm{Ind}_{H_n^B \otimes H_m^A}^{H_{n+m}^B}(L \otimes M)$ and $L \mapsto \mathrm{Res}_{H_i^B \otimes H_j^A}^{H_n^B} L$. Let θ be the automorphism of I given by $a \mapsto a^{-1}$. Then it induces an automorphism of U_I. Let U_I^θ be the θ-fixed point sets of U_I. Then the action of $\mathscr{O}(U_I) \simeq \mathbf{C} \otimes K_I^A$ on K_I^B descends to an action of $\mathscr{O}(U_I^\theta)$, as it follows from the following lemma.

Lemma 3.6. *For an irreducible H_n^B-module L and an irreducible H_m^A-module M, we have $\mu(L \otimes M) = \mu(L \otimes M^\theta)$, where M^θ is the H_m^A-module induced from M by the automorphism of H_m^A given by $X_i \mapsto X_{m+1-i}^{-1}$, $T_i \mapsto T_{m-i}$.*

Now we take the case

$$I = \{p_1^n \,;\, n \in \mathbf{Z}_{\mathrm{odd}}\}.$$

Assume that any of ± 1 and $\pm p_0$ is not contained in I. Let us define an automorphism θ of I by $a \mapsto a^{-1}$. The set I may be regarded as the set of vertices of a Dynkin diagram by (7). Let $\mathfrak{g}_I$ be the associated Lie algebra ($\mathfrak{g}_I$ is isomorphic to $\mathfrak{gl}_\infty$ if p_1 has an infinite order, and isomorphic to $A_\ell^{(1)}$ if p_1^2 is a primitive ℓ-th root of unity). Let $V_\theta(\lambda)$ be as in Proposition 2.5 with $\lambda = 0$.

Conjecture 3.7. (i) *$V_\theta(\lambda)$ has a crystal basis and a global basis.*

(ii) *$\mathbf{C} \otimes K_I^B$ is isomorphic to a specialization of $V_\theta(\lambda)$ at $q = 1$ as an $\mathscr{O}(U_I)$-module, and the elements of K_I^B associated with irreducible representations corresponds to the upper global basis of $V_\theta(\lambda)$ at $q = 1$.*

Note that (i) is nothing but Theorem 2.8 when p_1 is not a root of unity.

Let us take the case

$$I = \{p_0 p_1^{2n} \,;\, n \in \mathbf{Z}\} \cup \{p_0^{-1} p_1^{2n} \,;\, n \in \mathbf{Z}\}.$$

Assume that there exists no integer n such that $p_0^2 = p_1^{4n}$. It includes the case where $p_0 = p_1$ and $p_1^{2n} \neq 1$ for any $n \in \mathbf{Z}_{\mathrm{odd}}$. Let θ be the automorphism of I given by $\theta\colon a \mapsto a^{-1}$. Then θ has no fixed points. We regard I as the set of vertices of a Dynkin diagram by (7). Let $\mathfrak{g}_I$ be the associated Lie algebra. It is isomorphic to either $\mathfrak{gl}_\infty \oplus \mathfrak{gl}_\infty$, $\mathfrak{gl}_\infty$, $A_\ell^{(1)} \oplus A_\ell^{(1)}$ or $A_\ell^{(1)}$. Set $\lambda = \Lambda_{p_0} + \Lambda_{p_0^{-1}}$ (i.e. $(\alpha_i, \lambda) = \delta_{i,p_0} + \delta_{i,p_0^{-1}}$).

Conjecture 3.8. (i) *$V_\theta(\lambda)$ has a crystal basis and a global basis.*

(ii) *$\mathbf{C} \otimes K_I^B$ is isomorphic to a specialization of $V_\theta(\lambda)$ at $q = 1$ as an $\mathscr{O}(U_I)$-module, and the elements of K_I^B associated with irreducible representations corresponds to the upper global basis of $V_\theta(\lambda)$ at $q = 1$.*

In the both cases, we conjecture that, for an irreducible H_n^B-module M corresponding to an upper global basis $G^{\mathrm{up}}(b)$, $\dim M_a$ coincides with the value of $(\phi_\lambda, E_{a_1} \cdots E_{a_n} G^{\mathrm{up}}(b))$ at $q = 1$ for $a = (a_1, \ldots, a_n) \in I^n$.

Miemietz ([6]) introduced the operators $\tilde{e}_i$ and $\tilde{f}_i$ on the set of isomorphic classes of irreducible modules, similarly to the A type case, and studied their properties. We conjecture that they coincide with the operators $\tilde{E}_i$ and $\tilde{F}_i$ on $B_\theta(\lambda)$.

Remark 3.9. For an H_n^B-module N, we have

$$\mathrm{Res}_{H_{n+1}^A}^{H_{n+1}^B}\left(\mathrm{Ind}_{H_n^B \otimes H_1^A}^{H_{n+1}^B}(N \otimes \mathbf{C}_a)\right)$$
$$= \mathrm{Ind}_{H_n^A \otimes H_1^A}^{H_{n+1}^A}(N \otimes \mathbf{C}_a) + \mathrm{Ind}_{H_1^A \otimes H_n^A}^{H_{n+1}^A}(\mathbf{C}_{a^{-1}} \otimes N).$$

In the right-hand-side, the first term corresponds to the left multiplication of f_a and the last term corresponds to the right multiplication of $f_{a^{-1}}$. This leads the last formula in (3), which is the starting point of our study.

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