On the Kleiman-Mori cone

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Abstract: The Kleiman-Mori cone plays important roles in the birational geometry. In this paper, we construct complete varieties whose Kleiman-Mori cones have interesting properties. First, we construct a simple and explicit example of complete non-projective singular varieties for which Kleiman's ampleness criterion does not hold. More precisely, we construct a complete non-projective toric variety X and a line bundle L on X such that L is positive on $\overline{NE}(X) \setminus \{0\}$. Next, we construct complete singular varieties X with $NE(X) = N_1(X) \simeq \mathbf{R}^k$ for any k. These explicit examples seem to be missing in the literature.

Key words: Ampleness; projectivity; toric geometry; Kleiman-Mori cone.

1. Introduction. The Kleiman-Mori cone plays important roles in the birational geometry. In this paper, we construct complete varieties whose Kleiman-Mori cones have interesting properties. First, we construct a simple and explicit example of complete non-projective singular varieties for which Kleiman's ampleness criterion does not hold. More precisely, we construct a complete non-projective toric variety X and a line bundle L on X such that L is positive on $\overline{NE}(X) \setminus \{0\}$.

Definition 1.1. Let V be a complete algebraic scheme defined over an algebraically closed field k. We say that *Kleiman's ampleness criterion holds* for V if and only if the interior of the nef cone of V coincides with the ample cone of V.

Note that Kleiman's original statements are very sharp. We recommend the reader to see [6, Chapter IV §2 Theorems 1, 2]. Of course, our example is not "quasi-divisorial"in the sense of Kleiman (see [6, Chapter IV §2 Definition 4 and Theorem 2]). We do not repeat the definition of *quasi-divisorial* since we do not use it in this paper. Note that if X is projective or $\mathbf{Q}$-factorial then X is quasi-divisorial in the sense of Kleiman. Next, we construct complete singular varieties X with $NE(X) = N_1(X) \simeq \mathbf{R}^k$ for any k. We note that the condition $NE(X) = N_1(X)$ is equivalent to the following one: a line bundle L is nef if and only if L is numerically trivial. These explicit examples seem to be missing in the literature.

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We adopt the toric geometry to construct examples.

Notation. We use the basic notation of the toric geometry throughout this paper. Let $v_i \in N = \mathbf{Z}^3$ for $1 \le i \le k$. Then the symbol $\langle v_1, v_2, \dots, v_k\rangle$ denotes the cone $\mathbf{R}_{\ge 0}v_1 + \mathbf{R}_{\ge 0}v_2 + \cdots + \mathbf{R}_{\ge 0}v_k$ in $N_{\mathbf{R}}$.

2. Eikelberg's formula. In this section, we quickly review Eikelberg's formula, which we use in the following sections. For the proof, see [1].

Theorem 2.1 (cf. [1, Theorem 3.2]). *Let $X := X(\Delta)$ be a d-dimensional complete toric variety given by a fan Δ, $\Delta^{(d)} := \{\sigma \in \Delta \mid \dim \sigma = d\}$, $\Delta^{(1)} := \{\tau_1, \dots, \tau_n\}$ and n the number of the one-dimensional cones in Δ. Furthermore, let v_i be the primitive lattice point of $\tau_i \in \Delta^{(1)}$ and $D = \eta_1 V(\tau_1) + \cdots + \eta_n V(\tau_n)$ a torus invariant Cartier divisor such that $\eta_i \neq 0$ for all i. Define the space of all affine dependences*

$$\mathrm{AD}(\sigma) := \Big\{(\alpha_1, \dots, \alpha_n) \in \mathbf{Q}^n \;\Big|\; \sum_{\tau_j \subset \sigma} \alpha_j \frac{1}{\eta_j} v_j = 0, \quad \sum_{\tau_j \subset \sigma} \alpha_j = 0, \quad \text{and } \alpha_j = 0 \text{ if } \tau_j \not\subset \sigma \Big\}.$$

Then $\mathrm{Pic}\, X \simeq \mathbf{Z}^{n-d-\dim_{\mathbf{Q}} \sum_{\sigma \in \Delta^{(d)}} \mathrm{AD}(\sigma)}$.

3. On Kleiman's ampleness criterion. In this section, we construct explicit examples for which Kleiman's ampleness criterion does not hold. We think that the following example is the simplest one. It seems to be easy to construct a lot of singular toric varieties for which Kleiman's ampleness criterion does not hold. The reader can find many examples of singular toric 3-folds in [5]. He can easily check that Kleiman's ampleness criterion does not hold for X_6 in [5]. For the cone theorem of toric varieties, see [4, Theorem 4.1].

3.1 (Construction)**.** We fix $N = \mathbf{Z}^3$. We put

$$v_1 = (1,0,1), \quad v_2 = (0,1,1), \quad v_3 = (-1,-1,1),$$
$$v_4 = (1,0,-1), \quad v_5 = (0,1,-1), \quad v_6 = (-1,-1,-1).$$

We consider the following fans.

$$\Delta_P = \left\{ \begin{array}{l} \langle v_1, v_2, v_4\rangle, \quad \langle v_2, v_4, v_5\rangle, \langle v_2, v_3, v_5, v_6\rangle, \\ \langle v_1, v_3, v_4, v_6\rangle, \langle v_1, v_2, v_3\rangle, \quad \langle v_4, v_5, v_6\rangle, \\ \text{and their faces} \end{array} \right\},$$

and

$$\Delta_Q = \left\{ \begin{array}{l} \langle v_1, v_2, v_4, v_5\rangle, \langle v_2, v_3, v_5, v_6\rangle, \\ \langle v_1, v_3, v_4, v_6\rangle, \quad \langle v_1, v_2, v_3\rangle, \quad \langle v_4, v_5, v_6\rangle, \\ \text{and their faces} \end{array} \right\}.$$

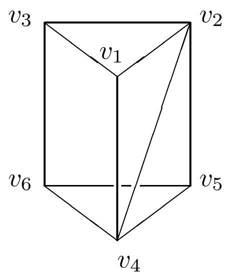

Fig. 1. Δ_P.

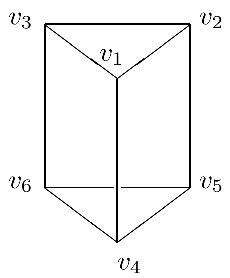

Fig. 2. Δ_Q.

Lemma 3.2. *We have the following properties*:

(i) $X_P := X(\Delta_P)$ *is a non-projective complete toric variety with* $\rho(X_P) = 1$,

(ii) $X_Q := X(\Delta_Q)$ *is a projective toric variety with* $\rho(X_Q) = 1$,

(iii) *there exists a toric birational morphism* $f_{PQ}: X_P \longrightarrow X_Q$, *which contracts a* $\mathbf{P}^1$ *on* X_P,

(iv) X_P *and* X_Q *have only canonical Gorenstein singularities, and*

(v)
$$\begin{array}{ccc} N_1(X_P) & \simeq & N_1(X_Q) \\ \cup & & \cup \\ NE(X_P) & \simeq & NE(X_Q). \end{array}$$

In particular, $NE(X_P) = \overline{NE}(X_P)$ *is a half line.*

Proof. It is easy to check that X_Q is projective and $\rho(X_Q) = 1$ (cf. [1, Example 3.5], Theorem 2.1). Assume that X_P is projective. Then there exists a strict upper convex support function h. We note that

$$v_1 + v_5 = v_2 + v_4,$$
$$v_2 + v_6 = v_3 + v_5,$$
$$v_3 + v_4 = v_1 + v_6.$$

Thus, we obtain

$$h(v_1) + h(v_5) < h(v_2) + h(v_4),$$
$$h(v_2) + h(v_6) = h(v_3) + h(v_5),$$
$$h(v_3) + h(v_4) = h(v_1) + h(v_6).$$

This implies that

$$\sum_{i=1}^{6} h(v_i) < \sum_{i=1}^{6} h(v_i).$$

It is a contradiction. Therefore, X_P is not projective. Thus f_{PQ} is not a projective morphism. So, $L \cdot C = 0$ for every $L \in \mathrm{Pic}(X_P)$, where $C \simeq \mathbf{P}^1$ is the exceptional locus of f_{PQ}. We note that the condition (ii) (b) in [6, p. 325 Theorem 1] does not hold. The other statements are easy to check. □

Let H be an ample Cartier divisor on X_Q and $D := (f_{PQ})^*H$. Then D is positive on $\overline{NE}(X_P) \setminus \{0\} = NE(X_P) \setminus \{0\}$. Thus, the interior of the nef cone of X_P is non-empty (cf. [6, p. 327 Proposition 2]). However, D is not ample on X_P. Therefore, Kleiman's ampleness criterion does not hold for X_P. Note that X_P is not projective nor quasi-divisorial in the sense of Kleiman (see [6, p. 326 Definition 4]).

Therefore, Kleiman's ampleness criterion does not hold for arbitrary complete toric varieties (see Proposition 3.7 below).

Corollary 3.3. *We put* $X := X_P \times \mathbf{P}^1 \times \cdots \times \mathbf{P}^1$. *Then we obtain complete non-projective singular toric varieties with* $\dim X \geq 4$ *for which Kleiman's ampleness criterion does not hold. Since every complete toric surface is* $\mathbf{Q}$*-factorial and projective, Kleiman's ampleness criterion always holds* *for toric surfaces.*

Remark 3.4. Let X be a complete $\mathbf{Q}$-factorial algebraic variety. Then it is not difficult to see that X is projective if $\rho(X) = 1$. We note that we can apply Kleiman's criterion since X is $\mathbf{Q}$-factorial. So, all we have to prove is that $\overline{NE}(X)$ is a half line. We assume that $NE(X) = \overline{NE}(X) \simeq \mathbf{R}$. Thus, there exist integral curves C and C' on X such that $C = \alpha C'$ in $\overline{NE}(X)$ with $\alpha < 0$. We can take effective divisors D and D' such that $D \cdot C > 0$ and $D' \cdot C' > 0$. Then, all the curves that intersect D properly are contained in D'. It is a contradiction. Therefore, we obtain that $\overline{NE}(X)$ is a half line. So, every effective Weil divisor is an ample $\mathbf{Q}$-Cartier divisor.

Remark 3.5. In [7, Chapter VI. Appendix 2.19.3 Exercise], Kollár pointed out that Kleiman's ampleness criterion does not hold for smooth proper algebraic spaces.

Remark 3.6. In [4, Theorem 4.1], we claim that $NE(X/Y)$ is strongly convex if $f : X \longrightarrow Y$ is projective. This is obvious. However, in the proof of Theorem 4.1 in [4], we say that it follows from Kleiman's criterion. Sorry, it is misleading.

We note the following ampleness criterion, which works for complete toric varieties with *arbitrary* singularities.

Proposition 3.7. *Let X be a complete toric variety and L a line bundle on X. Assume that $L \cdot C > 0$ for every torus invariant integral curve C on X. Then L is ample. In particular, X is projective.*

Proof. Since $NE(X)$ is spanned by the torus invariant curves on X, it is obvious that L is nef. This implies that L is generated by its global sections. Note that we can replace X (resp. L) with its toric resolution Y (resp. the pull-back of L on Y) in order to check the freeness of L. Thus, the proof of the freeness is easy. We consider the equivariant morphism $f := \Phi_{|mL|} : X \longrightarrow Y$ associated to the linear system $|mL|$, where m is a sufficiently large positive integer. Then we obtain that $mL = f^*H$ for a very ample line bundle H on Y. It is not difficult to see that $V := \{y \in Y \mid \dim f^{-1}(y) \geq 1\}$ is a torus invariant closed subset of Y. If V is not empty, then V contains a torus invariant point P of Y. We can find a torus invariant curve C in $f^{-1}(P)$. Then $mL \cdot C = f^*H \cdot C = 0$ by the projection formula. It is a contradiction. Therefore, f is finite. Thus, L is ample. □

4. Singular varieties with $NE(X) = N_1(X)$.

In this section, we construct complete singular toric varieties with $NE(X) = N_1(X) \simeq \mathbf{R}^k$ for any $k \geq 0$.

Remark 4.1. The condition $NE(X) = N_1(X)$ is equivalent to the following one: a line bundle L is nef if and only if L is numerically equivalent to zero.

Remark 4.2. Let X be a complete toric variety and L a line bundle on X. Then, it is well-known that L is numerically equivalent to zero if and only if it is trivial.

4.3 (Construction)**.** We fix $N = \mathbf{Z}^3$ and $M := \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z}) \simeq \mathbf{Z}^3$. We put

$$v_1 = (1,0,1), \quad v_2 = (0,1,1), \quad v_3 = (-1,-2,1),$$
$$v_4 = (1,0,-1), \quad v_5 = (0,1,-1), \quad v_6 = (-1,-1,-1),$$
$$v_7 = (0,0,-1).$$

First, we consider the following fan.

$$\Delta_A = \left\{ \begin{array}{l} \langle v_1, v_2, v_4, v_5 \rangle, \langle v_2, v_3, v_5, v_6 \rangle, \\ \langle v_1, v_3, v_4, v_6 \rangle, \quad \langle v_1, v_2, v_3 \rangle, \quad \langle v_4, v_5, v_6 \rangle, \\ \text{and their faces} \end{array} \right\}.$$

We put $X_A := X(\Delta_A)$ and $D_i := V(v_i)$ for every i. This example X_A is essentially the same as [1, Example 3.5]. We consider the principal divisor

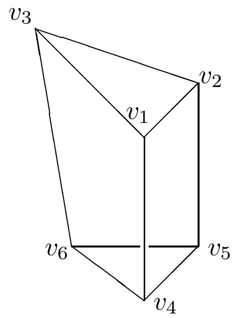

Fig. 3. Δ_A.

$$D = D_1 + D_2 + D_3 - D_4 - D_5 - D_6$$

that is associated to $m = (0,0,-1) \in M$. We put

$$\sigma_1 = \langle v_1, v_2, v_4, v_5 \rangle, \quad \sigma_2 = \langle v_2, v_3, v_5, v_6 \rangle,$$
$$\sigma_3 = \langle v_1, v_3, v_4, v_6 \rangle.$$

Then, all points in $\sum_{i=1}^{3} \mathrm{AD}(\sigma_i)$ are linear combinations of the row vectors of the matrix

$$\begin{pmatrix} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -3 & 2 \\ -2 & 0 & 1 & -1 & 0 & 2 \end{pmatrix}$$

which has rank 3. Note that we use D to define $\mathrm{AD}(\sigma_i)$ and that σ_i is not simplicial for every i and all the other 3-dimensional cones in Δ_A are simplicial. For the definition of $\mathrm{AD}(\sigma_i)$, see Theorem 2.1. Therefore, $\mathrm{Pic}\, X_A \simeq \mathbf{Z}^{6-3-3} = \{0\}$ by Theorem 2.1.

Next, we consider the following fan.

$$\Delta_B = \left\{ \begin{array}{l} \langle v_1, v_2, v_4, v_5\rangle, \langle v_2, v_3, v_5, v_6\rangle, \\ \langle v_1, v_3, v_4, v_6\rangle, \quad \langle v_1, v_2, v_3\rangle, \quad \langle v_4, v_5, v_7\rangle, \\ \quad \langle v_4, v_6, v_7\rangle, \qquad \langle v_5, v_6, v_7\rangle, \\ \text{and their faces} \end{array} \right\}.$$

We recommend the reader to draw the picture of Δ_B by himself. We put $X_B := X(\Delta_B)$. Then $X_B \longrightarrow X_A$ is the blow up along $\langle v_7\rangle$. We consider the principal divisor

$$D' = D_1 + D_2 + D_3 - D_4 - D_5 - D_6 - D_7$$

that is associated to $m = (0, 0, -1) \in M$. Then, all points in $\sum_{i=1}^{3} \mathrm{AD}(\sigma_i)$ are linear combinations of the row vectors of the matrix

$$\begin{pmatrix} 1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 & -3 & 2 & 0 \\ -2 & 0 & 1 & -1 & 0 & 2 & 0 \end{pmatrix}$$

which has rank 3. We note that we use D' to define $\mathrm{AD}(\sigma_i)$ and that σ_i is not simplicial for every i and all the other 3-dimensional cones in Δ_B are simplicial. Thus, we have $\mathrm{Pic}\, X_B \simeq \mathbf{Z}^{7-3-3} = \mathbf{Z}$ by Theorem 2.1.

Lemma 4.4. *D_7 is a Cartier divisor.*

Proof. It is because each 3-dimensional cone containing v_7 is non-singular. Thus, D_7 is Cartier. □

We put $C_1 := V(\langle v_4, v_5\rangle) \simeq \mathbf{P}^1$ and $C_2 := V(\langle v_4, v_7\rangle) \simeq \mathbf{P}^1$. The following lemma is a key property of this example.

Lemma 4.5. *$C_1 \cdot D_7 > 0$ and $C_2 \cdot D_7 < 0$. More precisely, $C_1 \cdot D_7 = 1$ and $C_2 \cdot D_7 = -3$. Therefore, D_7 is a generator of $\mathrm{Pic}\, X_B \simeq \mathbf{Z}$.*

Proof. It is obvious that $C_1 \cdot D_7 = 1 > 0$. Since $v_4 + v_5 + v_6 - 3v_7 = 0$, we have $C_2 \cdot D_7 = -3C_2 \cdot D_5 = -3 < 0$. □

Therefore, $NE(X_B) = N_1(X_B) \simeq \mathbf{R}$. In particular, X_B is not projective.

Corollary 4.6. *Let $X := X_A$. Then $NE(X) = N_1(X) = \{0\}$. Let $X := X_B \times X_B \times \cdots \times X_B$ be the k times product of X_B. Then $NE(X) = N_1(X) \simeq \mathbf{R}^k$.*

Proof. The first statement is obvious by the above construction. It is not difficult to see that $\mathrm{Pic}\, X \simeq \bigoplus_{i=1}^{k} p_i^* \mathrm{Pic}\, X_B$, where $p_i : X \longrightarrow X_B$ is the i-th projection. Thus, we can check that any nef line bundle on X is numerically trivial. Therefore, $NE(X) = N_1(X) \simeq \mathbf{R}^k$. □

In the above corollary, $\dim X = 3k$ if $NE(X) = N_1(X) \simeq \mathbf{R}^k$ for $k \geq 1$. We can construct 3-folds with $NE(X) = N_1(X) \simeq \mathbf{R}^k$ for every $k \geq 0$. We just note this fact in the next remark. The details are left to the reader.

Remark 4.7. We put $X_0 := X_A$, where X_A is in 4.3. We define primitive vectors $\{v_k\}$ inductively for $k \geq 7$ as follows: $\langle v_k\rangle = \langle v_4 + v_5 + v_{k-1}\rangle$. Let $X_k \longrightarrow X_{k-1}$ be the blow up along $\langle v_{k+6}\rangle$ for $k \geq 1$. Note that $X_1 = X_B$, where X_B is in 4.3. It is not difficult to see that $NE(X_k) = N_1(X_k) \simeq \mathbf{R}^k$ and $\dim X_k = 3$. We note that the numerical equivalence classes of $D_i = V(v_i)$'s for $7 \leq i \leq k+6$ form a basis of $N^1(X_k)$.

5. Miscellaneous comments.

In this section, we collect miscellaneous examples.

5.1 (Non-singular complete algebraic varieties)**.** After I circulated a preliminary version [3] of this paper, Sam Payne constructed beautiful examples (see [9]). These are counterexamples to Conjecture 4.5 in [3]. Let l be any integer such that $l \geq 11$. He constructed X_l with the following properties.

- X_l is a non-singular complete toric 3-fold,
- X_l has no nontrivial nef line bundles, that is, $NE(X_l) = N_1(X_l)$ (see Remark 4.1), and
- the Picard number $\rho(X_l) = l \geq 11$.

For the details, see [9]. I think that the following problem is still open.

Problem 5.2. Are there non-singular complete algebraic varieties X with $NE(X) = N_1(X)$ and $2 \leq \rho(X) \leq 10$?

Remark 5.3. Let X be a non-singular complete algebraic variety. Then $\rho(X) \geq 1$. It is obvious that X is projective when $\rho(X) = 1$ (see Remark 3.4). It does not necessarily hold for algebraic spaces (see Example 5.7 below).

Note the following remarks when we think Problem 5.2.

Remark 5.4. Let X be a complete normal (not necessarily $\mathbf{Q}$-factorial) variety. If there exists a proper surjective morphism $f : X \longrightarrow Y$ such that Y

is projective and $\dim Y \geq 1$, then it is obvious that $NE(X) \subset \overline{NE}(X) \subsetneq N_1(X)$.

Remark 5.5. If X is a complete (not necessarily $\mathbf{Q}$-factorial) toric variety with $NE(X) \subsetneq N_1(X)$, then there exists a nontrivial nef line bundle $\mathcal{L}$ on X. Thus, we obtain the toric morphism $f : X \longrightarrow Y := \mathrm{Proj} \bigoplus_{k \geq 0} H^0(X, \mathcal{L}^{\otimes k})$ such that $\mathcal{L} = f^*\mathcal{H}$, where $\mathcal{H}$ is an ample line bundle on Y. Note that $\dim Y \geq 1$.

5.6 (Algebraic space). The following example is more or less well-known to the experts. I learned it from S. Mori, who call it Hironaka's example.

Example 5.7. Let $Q \simeq \mathbf{P}^1 \times \mathbf{P}^1$ be a non-singular quadric surface in $\mathbf{P}^3_{\mathbf{C}}$. We take a non-singular $(3, d)$-curve C in Q, where $d \in \mathbf{Z}_{>0}$. That is, $\mathcal{O}_Q(C) \simeq p_1^*\mathcal{O}_{\mathbf{P}^1}(3) \otimes p_2^*\mathcal{O}_{\mathbf{P}^1}(d)$, where p_1 (resp. p_2) is the first (resp. second) projection from Q to $\mathbf{P}^1$. Let f_1 (resp. f_2) be a fiber of $p_2 : Q \longrightarrow \mathbf{P}^1$ (resp. p_1). We take the blow up $\pi : X \longrightarrow \mathbf{P}^3$ along C. Let Q' be the strict transform of Q and E the exceptional divisor of π. Then $Q' = f^*Q - E$. Let f_i' be the strict transform of f_i for $i = 1, 2$. We have

$$Q' \cdot f_1' = f^*Q \cdot f_1' - E \cdot f_1' = Q \cdot f_1 - 3 = 2 - 3 = -1.$$

Thus we can blow down X to Y along the ruling $p_2 : Q' \simeq Q \longrightarrow \mathbf{P}^1$ (cf. [8, Main Theorem] and [2]). Note that Y is a compact Moishezon manifold. The Kleiman-Mori cone $\overline{NE}(X)$ is spanned by 2 rays R and R'. We note that X is non-singular projective and $\rho(X) = 2$. Let l be a fiber of $\pi : X \longrightarrow \mathbf{P}^3$. Then, one ray R is spanned by the numerical equivalence class of l. We put $\mathcal{L} := \pi^*\mathcal{O}_{\mathbf{P}^3}(1)$. Then $\mathcal{L}$ is non-negative on $\overline{NE}(X)$ and $R = (\mathcal{L} = 0) \cap \overline{NE}(X)$. We have the following intersection numbers.

$$\mathcal{L} \cdot l = 0, \quad \mathcal{L} \cdot f_1' = \mathcal{L} \cdot f_2' = 2,$$
$$E \cdot l = -1, \quad E \cdot f_1' = 3, \quad E \cdot f_2' = d.$$

From now on, we assume $d \geq 4$. We can write $f_1' = af_2' + bl$ in $N_1(X)$ for $a, b \in \mathbf{R}$. Thus we can easily check that $a = 1$, $b = d - 3 > 0$. Therefore, the numerical equivalence class of f_1' is in the interior of the cone spanned by the numerical equivalence classes of f_2' and l. Thus, we have $NE(Y) = \overline{NE}(Y) = N_1(Y) \simeq \mathbf{R}$. Therefore, Y is a non-singular complete algebraic space with $\rho(X) = 1$. Note that Y is not a scheme (cf. Remark 3.4).

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