# Conformally flat metrics and $S^1$-fibration

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**Abstract:** Characterization of conformally flat bundle metric on $S^1$-principal bundle is studied. It is shown further that there are infinitely many compact conformally flat $S^1$-principal bundles which are new examples, besides the Hopf fibration.

**Key words:** Conformally flat bundle metric; $S^1$-bundle; Yang-Mills connection.

**1.** We will consider conformal flatness of smooth manifolds which carry an $S^1$-fibration. To formulate the problem we let $\pi : P \longrightarrow M$ be an $S^1$-principal bundle with a connection $\gamma$ over a Riemannian $n$-manifold $(M, h)$, $n \geq 3$. Then $\gamma$ provides a metric $g$ on $P$, called bundle metric, as $g = \pi^*h + \gamma \otimes \gamma$, yielding the splitting of tangent space $T_uP$ as the vertical and the horizontal subspaces; $T_uP = V_u \oplus H_u$, $u \in P$ such that $V_u \perp H_u$ and $g|_{H_u} = \pi^*h|_{T_xM}$ at $x = \pi(u)$.

Classically well known examples of conformally flat manifold carrying $S^1$-fibration are the Hopf fibration $S^{2n+1} \longrightarrow \mathbf{C}P^n$ and its $\mathbf{Z}_k$-quotient $S^{2n+1}/\mathbf{Z}_k \longrightarrow \mathbf{C}P^n$ and their Riemannian product either with the flat line $\mathbf{R}$ or a real hyperbolic space with curvature of opposite sign (see [3]).

The aim of this note is to report that we establish characterizing theorems which are almost complete and that we present, from these theorems, countably many conformally flat $S^1$-principal bundles over $\mathbf{C}P^1 \times \cdots \times \mathbf{C}P^1$, as new examples. The complete characterization under our program and the details of our theorems presented in this note will be published elsewhere ([1]).

We assume that the curvature form $\Gamma$ of $\gamma$ is harmonic, that is, $\gamma$ is Yang-Mills and that $(M, h)$ is simply connected, connected and complete.

The following is a key theorem for investigating the conformal flatness of bundle metric.

**Theorem A** ([1, 2, 4]). *Let $\pi : P \longrightarrow M$ be an $S^1$-principal bundle with a non-flat Yang-Mills connection $\gamma$ over an oriented Riemannian $n$-manifold $(M, h)$. Then $g = \pi^*h + \gamma \otimes \gamma$ on $P$ is conformally flat if and only if the following hold over $M$*

$$W = -\frac{1}{(n-2)(n-1)} T \owedge h - \frac{1}{2(n-1)} H \owedge h + \frac{1}{4}\Gamma \ovee \Gamma - \frac{3C}{8n(n-1)} h \owedge h,$$

$$H = \frac{4}{n+1} T \quad \text{and}$$

$$\nabla\Gamma = 0.$$

*Here $W$ denotes the Weyl curvature tensor, $T$ the trace free Ricci tensor of $(M, h)$, respectively and $H$ is the trace free symmetric tensor, defined $H_{ij} = B_{ij} - \frac{C}{n}h_{ij}$ and $B_{ij} = \sum_s \Gamma_{si}\Gamma_{sj}$ and $C = \sum_s B_{ss} = \frac{1}{2}|\Gamma|^2$.*

Therefore, if a bundle metric is conformally flat, $(M, h)$ must be locally symmetric, since $\Gamma$ and hence $T$ is parallel so that $W$ is parallel. Define the distribution $\mathcal{D} = \{X \in TM \mid i_X\Gamma = 0\}$ and its orthogonal complement $\mathcal{D}^\perp$ on $M$.

**Lemma** ([1, 2]). *Assume that $\Gamma$ is parallel. Then, $\mathcal{D}$ and $\mathcal{D}^\perp$ are of constant rank, and integrable and invariant under the holonomy action.*

The following four possibilities for $\mathcal{D}$ and $\mathcal{D}^\perp$ may occur:

**Case 1.** $\mathcal{D} = \{0\}$, and $\mathcal{D}^\perp$ is holonomy-irreducible, in other words, the homogeneous holonomy group $\Psi$ acts on $\mathcal{D}^\perp$ irreducibly.

**Case 2.** $\mathcal{D} \neq \{0\}$ and $\mathcal{D}^\perp$ is holonomy-irreducible.

**Case 3.** $\mathcal{D} = \{0\}$ and $\mathcal{D}^\perp$ is not holonomy-irreducible, and

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Remark that $\mathcal{D}^\perp$ is non-trivial, if $\gamma$ is not flat, and that $\mathcal{D} = \{0\}$ if and only if $\Gamma$ is non-degenerate as a 2-form on $M$.

**2.** Cases 1 and 2 can be completely investigated as the following two theorems.

**Theorem B** ([1]). *$\pi : P \longrightarrow M$ be an $S^1$-principal bundle over $M$ with a non-flat Yang-Mills connection $\gamma$. Assume Case 1, namely, that $(M, h)$ is irreducible in the sense of de Rham and $\Gamma$ is a non-degenerate 2-form on $M$.*

*If $g = \pi^* h + \gamma \otimes \gamma$ is conformally flat, then $(M, h)$ and $(P, g)$ are isometric to $(\mathbf{C}P^m, h_{FS})$ and $(S^{2m+1}/\mathbf{Z}_k, g_S)$, respectively.*

*Furthermore, $P \longrightarrow M$ is isomorphic to $S^{2m+1}/\mathbf{Z}_k \longrightarrow \mathbf{C}P^m$.*

*Here $k \in \mathbf{Z}$ is the first Chern number of the associated complex line bundle $\mathcal{L}_P$ evaluated by the positive generator of $H_2(\mathbf{C}P^m, \mathbf{Z})$ and $h_{FS} = h_{FS}\left(\frac{4}{k^2}\right)$ is the Fubini-Study metric on $\mathbf{C}P^m$ of constant holomorphic curvature $\frac{4}{k^2}$ and $g_S = g_S\left(\frac{1}{k^2}\right)$ is the standard metric of constant sectional curvature $\frac{1}{k^2}$.*

*Moreover $\mathbf{Z}_k \cong \left\{e^{\frac{2\pi a}{k} i} \mid a = 1, \ldots, |k|\right\}$ acts canonically on $S^{2m+1} \subset \mathbf{C}^{m+1}$ and $S^{2m+1}/\mathbf{Z}_k$ is the $\mathbf{Z}_k$-quotient of $S^{2m+1}$.*

**Theorem C** ([1]). *In Case 2, the bundle $(P, g)$ is isometric either to*

$$(\mathbf{R}, dt^2) \times \left(S^{2m+1}/\mathbf{Z}_k, g_S\left(\frac{1}{k^2}\right)\right),$$

*or*

$$(H^\ell, h_{HYP}) \times \left(S^{2m+1}/\mathbf{Z}_k, g_S\left(\frac{1}{k^2}\right)\right),$$

*and the base manifold $(M, h)$ is isometric either to*

$$(\mathbf{R}, dt^2) \times \left(\mathbf{C}P^m, h_{FS}\left(\frac{4}{k^2}\right)\right),$$

*or*

$$\left(H^\ell, h_{HYP}\left(-\frac{1}{k^2}\right)\right) \times \left(\mathbf{C}P^m, h_{FS}\left(\frac{4}{k^2}\right)\right),$$

*respectively.*

*Moreover, $P \longrightarrow M$ is isomorphic either to*

$$\mathbf{R} \times S^{2m+1}/\mathbf{Z}_k \longrightarrow \mathbf{R} \times \mathbf{C}P^m,$$
$$(t, (z)_k) \longmapsto (t, [z]),$$

*or*

$$H^\ell \times S^{2m+1}/\mathbf{Z}_k \longrightarrow H^\ell \times \mathbf{C}P^m,$$
$$(q, (z)_k) \longmapsto (q, [z]),$$

*respectively, where the projections are canonical ones.*

These theorems indicate that conformally flat $S^1$-principal bundle metrics of **cases 1 and 2** are only the classically known examples.

The possibility of **case 4** is excluded by a straightforward calculation. See [1] for the detail.

**3.** To investigate **case 3** we discuss the following six types that cover all possibilities of $M$ which occur in case 3:

**Type 1.** $M = \mathbf{C}P^{m_1} \times \cdots \times \mathbf{C}P^{m_k}$,
$n = 2(m_1 + \cdots + m_k)$,

**Type 2.** $M = \mathbf{C}H^1 \times \cdots \times \mathbf{C}H^1$ ($\ell$-times),
$n = 2\ell$,

**Type 3.** $M = \mathbf{C}^1 \times \underbrace{\mathbf{C}H^1 \times \cdots \times \mathbf{C}H^1}_{\ell \text{ times}}$,
$n = 2(\ell + 1), \quad \ell \geq 1$,

**Type 4.** $M = \mathbf{C}^1 \times \underbrace{\mathbf{C}H^1 \times \cdots \times \mathbf{C}H^1}_{\ell \text{ times}} \times \mathbf{C}P^{m_1} \times \cdots \times \mathbf{C}P^{m_k}$,
$n = 2(\ell + 1) + 2(m_1 + \cdots + m_k), \ \ell \geq 1$,

**Type 5.** $M = \mathbf{C}^1 \times \mathbf{C}P^{m_1} \times \cdots \times \mathbf{C}P^{m_k}$,
$n = 2 + 2(m_1 + \cdots + m_k)$,

**Type 6.** $M = \underbrace{\mathbf{C}H^1 \times \cdots \times \mathbf{C}H^1}_{\ell \text{ times}} \times \mathbf{C}P^{m_1} \times \cdots \times \mathbf{C}P^{m_k}$,
$n = 2\ell + 2(m_1 + \cdots + m_k)$.

We have then for the first type

**Theorem D** ([1]). *Let $P \longrightarrow M$ be an $S^1$-principal bundle over $M$ and $M = \mathbf{C}P^{m_1} \times \cdots \times \mathbf{C}P^{m_k}$ be the de Rham decomposition of $M$. If a bundle metric on $P$ associated to a non-flat Yang-Mills connection is conformally flat, then $M$ is written as*

$$M = \underbrace{\mathbf{C}P^m \times \cdots \times \mathbf{C}P^m}_{\ell \text{ times}} \times \underbrace{\mathbf{C}P^1 \times \cdots \times \mathbf{C}P^1}_{\ell' \text{ times}}$$

*where $m$ is a positive integer. Furthermore*

(1) *For $\ell \geq 2$, $m \geq 2$ and $\ell' = 0$, a product manifold $M = \mathbf{C}P^m \times \cdots \times \mathbf{C}P^m$ ($\ell$-times) is eliminated from possibility of $M$ of Type 1.*

(2) *A Riemannian product manifold $M = M_1 \times M_2$, where $M_1 = \mathbf{C}P^m \times \cdots \times \mathbf{C}P^m$ ($\ell$-times), $\ell \geq 1$, $m \geq 2$ and $M_2 = \mathbf{C}P^1 \times \cdots \times \mathbf{C}P^1$ ($\ell'$-times), $\ell' \geq 1$ is eliminated from possibility of $M$ of Type 1.*

(3) *A product manifold $M = \mathbf{C}P^1 \times \cdots \times \mathbf{C}P^1$ ($\ell'$ times), $\ell' \geq 2$, $n = 2\ell'$ is not eliminated from pos-*

*sibility of a base manifold of Type* 1*, that is, there is an* $S^1$*-principal bundle over* $M$ *whose bundle metric is conformally flat.*

**Theorem E** ([1]). *Types* 2, 3, 4 *and* 5 *can be eliminated from the base manifold possibility.*

Type 6 remains to be solved. On the other hand, from (3) of Theorem D, there are countably many new examples of conformally flat $S^1$-principal bundle $P$ over $M = \mathbf{C}P^1 \times \cdots \times \mathbf{C}P^1$. These examples are then also equipped with Sasakian structure from theorem of Boothby-Wang-Hatakeyama, since $\Gamma$ is non-degenerate. The bundle $P$ here is a unitary frame bundle associated with the tensor product $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_{\ell'}$, where each $\mathcal{L}_i$, $i = 1, \ldots, \ell'$, is the pull back of certain Hermitian line bundle $\mathcal{L}$ over $\mathbf{C}P^1$ by the projection to the $i$-factor. See [1] for their precise construction. Also (3) of Theorem D improves Theorem 1.1 in [5] as follows:

**Theorem F** ([1]). *Let* $\pi : P \longrightarrow M$ *be an* $S^1$*-principal bundle over a connected oriented Riemannian* 4*-manifold* $(M, h)$ *and* $\gamma$ *a non-flat self-dual connection on* $P$*. If the bundle metric is conformally flat, then either*

(1) $(M, \frac{1}{24}\sigma\ h)$ *is locally isometric and biholomorphic to a domain* $D$ *of* $\mathbf{C}P^2$ *with the Fubini-Study metric and* $(P, g)$ *is of positive constant curvature* $\frac{1}{24}\sigma$*, and* $P \longrightarrow M$ *is a portion of the Hopf fibration, or*

(2) $(M, h)$ *is locally isometric and biholomorphic to a domain of* $\mathbf{C}P^1 \times \mathbf{C}P^1$ *with the standard product metric on* $\mathbf{C}P^1$ *and* $P$ *is the unitary frame bundle associated with the complex Hermitian line bundle* $\mathcal{L}_1 \otimes \mathcal{L}_2$ *over* $M$*.*

## References


[ 1 ] M. Itoh, N. Nakada and T. Satou, $S^1$-Fibration and the conformal flatness, (2004). (In preparation).

[ 2 ] M. Itoh, T. Satou, Circle bundle metric and the conformal flatness, (1998). (Preprint).

[ 3 ] J. Lafontaine, Conformal geometry from the Riemannian viewpoint, in *Conformal geometry* (*Bonn, 1985/1986*), 65–92, Aspects Math. E12, Vieweg, Braunschweig, 1998.

[ 4 ] T. Satou, Conformal flatness and self-duality of circle bundle metrics, Doctor thesis, Univ. of Tsukuba, (1998).

[ 5 ] T. Satou, Conformal flatness of circle bundle metric, Tsukuba J. Math. **22** (1998), no. 2, 349–355.