Valiron, Nevanlinna and Picard exceptional sets of iterations of rational functions

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Abstract: For every rational function of degree more than one, there exists a transcendental meromorphic solution of the Schröder equation. By Yanagihara and Eremenko-Sodin, it is known that the Valiron, Nevanlinna and Picard exceptional sets of this solution are all same.

As an analogue of this result, we show that all the Valiron, Nevanlinna and Picard exceptional sets of iterations of a rational function of degree more than one are also same. As a corollary, the equidistribution theorem in complex dynamics follows.

Key words: Schröder equation; Valiron exceptional set; complex dynamics; equidistribution.

1. Introduction. Let f be a rational function, i.e., a holomorphic endomorphism of the Riemann sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Assume that the degree $d := \deg f$ is more than one, and denote the k times iteration of f by f^k for $k \in \mathbf{N}$. We call $a \in \hat{\mathbf{C}}$ a *Picard exceptional value* of $\{f^k\}_{k\in\mathbf{N}}$ if

$$\# \bigcup_{k\in\mathbf{N}} f^{-k}(a) < \infty.$$

The *Picard exceptional set* $E(\{f^k\})$ is defined by that of all such points. It is well known that every point of $E(\{f^k\})$ is periodic of period at most two and critical of order $d-1$. In particular, $E(\{f^k\})$ contains at most two points (cf. [7]).

It is well known that for some $n \in \mathbf{N}$ and some $\lambda \in \mathbf{C}$ with $|\lambda| > 1$, the *Schröder equation*

$$h \circ \lambda = f^n \circ h \tag{1}$$

has a transcendental meromorphic solution h with $h'(0) \neq 0$. The value distribution of the solution h is studied by many authors. For example,

Theorem 1.1 (Yanagihara [10], Eremenko-Sodin [2]). *For the above solution h,*

$$E(\{f^k\}) = E_P(h) = E_N(h) = E_V(h),$$

where $E_P(h)$, $E_N(h)$ and $E_V(h)$ are the Picard, Nevanlinna, and Valiron exceptional sets of the transcendental meromorphic solution h (cf. [8]) respectively.

Remark. See also Ishizaki-Yanagihara's generalization of it ([4]). They also studied the value distribution of h in angular domains, and determined the *Borel and Julia directions* of h ([5]).

We note that f also acts on the space of all regular measures ((1, 1)-currents of order 0) on $\hat{\mathbf{C}}$ as the pullback operator f^*. In particular, for the Dirac measure δ_a at the value $a \in \hat{\mathbf{C}}$, $f^*\delta_a/d$ characterizes the averaged distribution of roots of the equation $f = a$. The *mean proximity* of f with respect to $a \in \hat{\mathbf{C}}$ is defined by

$$m(a, f) := \int_{\hat{\mathbf{C}}} \log \frac{1}{[a, f(w)]} \, \mathrm{d}\sigma(w),$$

where σ is the spherical area measure on $\hat{\mathbf{C}}$ normalized as $\sigma(\hat{\mathbf{C}}) = 1$ and $[z, w]$ the chordal distance between $z, w \in \hat{\mathbf{C}}$ normalized as $[0, \infty] = 1$. For a sequence $\{f_k\}_{k\in\mathbf{N}}$ of rational functions with increasing degrees $d_k := \deg f_k$, the *Valiron* and *Nevanlinna defects* are defined as

$$\delta_V(a; \{f_k\}) := \limsup_{k\to\infty} \frac{m(a, f_k)}{d_k} \quad \text{and}$$
$$\delta_N(a; \{f_k\}) := \liminf_{k\to\infty} \frac{m(a, f_k)}{d_k}$$

respectively, and the *Valiron* and *Nevanlinna exceptional sets* $E_V(\{f_k\})$ and $E_N(\{f_k\})$ by those of all points with non-zero Valiron and Nevanlinna defects respectively.

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Theorem 1.2 (Sodin [9])**.** *For every regular probability measure μ with no atom in $E_V(\{f_k\})$,*

$$\lim_{k\to\infty}\frac{(f_k)^*(\sigma-\mu)}{d_k}=0 \quad (\textit{weak}).$$

In [9], Sodin also estimates the Hausdorff measures of $E_V(\{f_k\})$.

Focusing on $\{f^k\}$, we now state our result in this paper, which is an analogue of Theorem 1.1:

Theorem (All exceptional sets are same.)**.** *Let f be a rational function of degree more than one. Then*

$$E(\{f^k\})=E_N(\{f^k\})=E_V(\{f^k\}).$$

In particular, they consist of at most two points.

From Theorem 1.2 and Theorem, the following holds:

Corollary 1.1 ([1], [6], and [3])**.** *There exists a regular probability measure μ_f such that for every regular probability measure μ on $\hat{\mathbf{C}}$ with $\mu(E(\{f^k\}))=0$,*

$$(2)\qquad \lim_{k\to\infty}\frac{(f^k)^*\mu}{d^k}=\mu_f \quad (\textit{weak}).$$

2. Proof of Theorem. Let g be a meromorphic function on $\mathbf{C}$. The *Picard exceptional set* $E_P(g)$ is defined by that of all such points $a\in\hat{\mathbf{C}}$ as

$$\#(g^{-1}(a))<\infty.$$

The (Shimizu-Ahlfors) *characteristic function* is defined by

$$T(r,g):=\int_0^r\frac{\mathrm{d}t}{t}\int_{\{z\in\hat{\mathbf{C}};|z|\le t\}}g^*\mathrm{d}\sigma \quad (r\ge 0),$$

and for $a\in\hat{\mathbf{C}}$, the *proximity function* and the *counting function* are defined by

$$m(r,a,g):=\int_0^{2\pi}\log\frac{1}{[a,g(re^{i\theta})]}\frac{\mathrm{d}\theta}{2\pi}, \quad \text{and}$$

$$N(r,a,g):=\int_0^r\frac{n(t,a,g)-n(0,a,g)}{t}\,\mathrm{d}t + n(0,a,g)\log r \quad (r\ge 0)$$

respectively, where $n(t,a,g):=\int_{\{z\in\mathbf{C};|z|\le t\}}g^*\delta_a$ for $t\ge 0$.

The *Valiron* and *Nevanlinna defects* of g at the value $a\in\hat{\mathbf{C}}$ are defined by

$$\delta_V(a;g):=\limsup_{r\to\infty}\frac{m(r,a,g)}{T(r,g)} \quad \text{and}$$

$$\delta_N(a;g):=\liminf_{r\to\infty}\frac{m(r,a,g)}{T(r,g)}$$

respectively, and the *Valiron* and *Nevanlinna exceptional sets* $E_V(g)$ and $E_N(g)$ by those of all points with non-zero Valiron and Nevanlinna defects respectively.

The *first main theorem* in the Shimizu-Ahlfors form [8] is the following:

$$(3)\quad m(r,a,g)+N(r,a,g)-T(r,g)=C_a(g) := \lim_{z\to 0}\log\frac{|z|^{n(0,a,g)}}{[g(z),a]}<\infty \quad (a\in\hat{\mathbf{C}},\ r\ge 0).$$

Let f be a rational function of the degree $d>1$. Without loss of generality, we assume that (1) has a transcendental meromorphic solution h for some $\lambda\in\mathbf{C}$ with $|\lambda|>1$ and $n=1$.

Let $\mathbf{D}_r:=\{z\in\hat{\mathbf{C}};|z|<r\}$ and $m_2=\mathrm{d}x\mathrm{d}y$ the planer area measure. The following is a corollary of the Böttcher theorem (cf. [7]):

Lemma 2.1. *For an $a\in E(\{f^k\})$ fixed by f, there exists a conformal map $z=\phi(w)$ from $\mathbf{D}_r$ ($r\in(0,1)$) into $\hat{\mathbf{C}}$ such that $\phi(0)=a$ and $f(\phi(w))=\phi(w^d)$ on $\mathbf{D}_r$, hence $f^k(\phi(w))=\phi(w^{d^k})$ there for every $k\in\mathbf{N}$.*

Corollary 2.1. $E(\{f^k\})\subset E_N(\{f^k\})$.

Proof. Let $a\in E(\{f^k\})$. Without loss of generality, we assume that a is fixed by f. Choose such a conformal map $z=\phi(w)$ on $\mathbf{D}_r$ as in Lemma 2.1. By a uniform distortion of ϕ on $\mathbf{D}_{r/2}$ by the Koebe theorem,

$$\frac{1}{d^k}\int_{\phi(\mathbf{D}_{r/2})}\log\frac{1}{[f^k(z),a]}\,\mathrm{d}\sigma(z) \ge M\int_{\mathbf{D}_{r/2}}\log\frac{1}{|w|}\,\mathrm{d}m_2(w)+o(1),$$

where $M:=\inf_{\mathbf{D}_{r/2}}(\phi^*\sigma/m_2)>0$. Hence $\delta_N(a;\{f^k\})>0$. □

Without loss of generality, we assume that $h(0)=0$, which also implies $f(0)=0$.

Lemma 2.2. *There exists a $C>0$ such that for every $k\in\mathbf{N}$ and every $t>0$, $T(|\lambda|^k t,h)\le d^k(T(t,h)+C)$.*

Proof. Let $\pi:\mathbf{C}^2-O\to\hat{\mathbf{C}}$ be the canonical projection which maps $Z=(z_0,z_1)$ to z_1/z_0 when $z_0\ne 0$. Here O is the origin in $\mathbf{C}^2$. There exists a homogeneous polynomial map $F=(F_0,F_1):\mathbf{C}^2\to\mathbf{C}^2$ of degree d such that $F(Z)=0$ if and only if $Z=O$, and $\pi\circ F=f\circ\pi$ on $\mathbf{C}^2-O$.

Let $\|\cdot\|$ be the Euclidean norm on $\mathbf{C}^2$. Since $f(0)=0$, it follows that $F_1(1,0)=0$ and $\|F(1,0)\|=|F_0(1,0)|$. There exist $C_1, C_2>0$ such that on $\{Z\in\mathbf{C}^2; \|Z\|=1\}$, it holds that $C_1\le\|F\|\le C_2$. Without loss of generality, we assume $C_2=1$. Then it holds that on $\{Z\in\mathbf{C}^2;\|Z\|=1\}$,

$$C_1^{1/(d-1)}\le\|F^k\|^{1/d^k}\le 1. \tag{4}$$

For $i=0,1$, define a $\phi_i:\hat{\mathbf{C}}\to\mathbf{C}$ as

$$\phi_i(z):=\begin{cases}F_i((1,h(z))/\|(1,h(z))\|) & \text{if } h(z)\ne\infty,\\ F_i((0,1)) & \text{otherwise.}\end{cases}$$

Then $f\circ h(z)=\phi_1(z)/\phi_0(z)$.

Put $\mathrm{d}=\partial+\overline{\partial}$ and $\mathrm{d}^c=(i/(2\pi))(\overline{\partial}-\partial)$. It holds that

$$\mathrm{dd}^c\log|\phi_0|=(f\circ h)^*\delta_\infty-d\cdot h^*\sigma$$

as currents. Hence for every $t\ge 0$, it follows that

$$\begin{aligned}&N(t,\infty,f\circ h)-dT(t,h)\\&=\int_0^{2\pi}\log|\phi_0(te^{i\theta})|\frac{\mathrm{d}\theta}{2\pi}-\log|\phi_0(0)|\\&\quad\text{(by } n(0,\infty,f\circ h)=0 \text{ and the Jensen formula)}\\&\le-\int_0^{2\pi}\log^+\frac{|\phi_1(te^{i\theta})|}{|\phi_0(te^{i\theta})|}\frac{\mathrm{d}\theta}{2\pi}-\log\|F(1,0)\|\\&\quad\text{(by } |\phi_i|\le 1 \text{ and } |\phi_0(0)|=|F_0(1,0)|=\|F(1,0)\|)\\&\le-(m(t,\infty,f\circ h)-\log\sqrt{2})-\log\|F(1,0)\|\end{aligned}$$

by $f\circ h(z)=\phi_1(z)/\phi_0(z)$ and $\log\sqrt{1+x^2}\le\log^+x+\log\sqrt{2}$ for $x\ge 0$. Hence, since $C_\infty(f\circ h)=\log(1/[(f\circ h)(0),\infty])=\log(1/[0,\infty])=0$, (3) implies that

$$\begin{aligned}T(|\lambda|t,h)&=T(t,h\circ\lambda)=T(t,f\circ h)\\&\le dT(t,h)+\log\frac{\sqrt{2}}{\|F(1,0)\|}\\&=d\left(T(t,h)+\log\frac{\sqrt{2}^{1/d}}{\|F(1,0)\|^{1/d}}\right).\end{aligned}$$

Applying the above argument and (4) to f^k (and F^k) for each $k\in\mathbf{N}$, we have

$$T(|\lambda|^k t,h)\le d^k\left(T(t,h)+\log\frac{\sqrt{2}^{1/d^k}}{C_1^{1/(d-1)}}\right).$$

□

Lemma 2.3. $E_V(\{f^k\})\subset E_V(h)$.

Proof. Let $a\in\hat{\mathbf{C}}-E_V(h)$. When $E(\{f^k\})$ is empty, put $U=\emptyset$. Otherwise, Lemma 2.1 implies that there exists an open neighborhood U of $E(\{f^k\})(=E_V(h)=E_P(h))$ such that $\overline{U}\subset\hat{\mathbf{C}}-\{a\}$ and $f(U)\subset U$. Choosing such an $r>0$ that $h(\mathbf{D}_r)\supset\hat{\mathbf{C}}-U$, we have:

$$\begin{aligned}&\frac{1}{d^k}\int_{\hat{\mathbf{C}}}\log\frac{1}{[f^k,a]}\mathrm{d}\sigma\\&\le\frac{1}{d^k}\left(\int_U+\int_{h(\mathbf{D}_r)}\right)\log\frac{1}{[f^k,a]}\mathrm{d}\sigma\\&\le\frac{1}{d^k}\log\frac{1}{[U,a]}+\frac{1}{d^k}\int_{\mathbf{D}_r}\log\frac{1}{[f^k\circ h,a]}h^*\mathrm{d}\sigma\\&\le\frac{1}{d^k}\log\frac{1}{[U,a]}+\frac{1}{d^k}\int_{\mathbf{D}_r}\log\frac{1}{[h\circ\lambda^k,a]}M_r\mathrm{d}m_2\\&\quad\text{(by (1))}\\&\le\frac{1}{d^k}\log\frac{1}{[U,a]}+M_r\int_0^r\frac{m(|\lambda^k|t,a,h)}{d^k}t\mathrm{d}t\\&\quad\text{(by the Fubini theorem),}\end{aligned}$$

where $M_r:=\sup_{\mathbf{D}_r}(h^*\sigma/m_2)<\infty$. By Lemma 2.2, for every $t\in[0,r]$, it follows that

$$\begin{aligned}0&\le\limsup_{k\to\infty}\frac{m(|\lambda^k|t,a,h)}{d^k}t\\&\le\limsup_{k\to\infty}\frac{m(|\lambda^k|t,a,h)}{T(|\lambda^k|t,h)}(T(t,h)+C)t\\&\le\delta_V(a;h)(T(t,h)+C)t=0.\end{aligned}$$

Furthermore, by (3) and Lemma 2.2, it holds that

$$\begin{aligned}0&\le\frac{m(|\lambda^k|t,a,h)}{d^k}t\\&\le\frac{T(|\lambda^k|t,h)+C_a(h)-n(0,a,h)\log(|\lambda^k|t)}{d^k}t\\&\le t(T(t,h)+C)+|C_a(h)|t+n(0,a,h)t\log^+(1/t),\end{aligned}$$

which is independent of $k\in\mathbf{N}$ and integrable on $[0,r]$. Hence by the dominated convergence theorem, $0\le\delta_V(a;\{f^k\})\le 0+M_r\cdot 0=0$. □

Gathering Lemmas 2.1 and 2.3 and Theorem 1.1, we conclude that

$$\begin{aligned}&E(\{f^k\})\subset E_N(\{f^k\})\subset E_V(\{f^k\})\subset E_V(h)\\&=E(\{f^k\}).\end{aligned}$$

Now completed is the proof of Theorem. □

3. A proof of Corollary 1.1. It is enough to show the weak convergence of $\{(f^k)^*\sigma/d^k\}$, which is well known. For reader's convenience, we include a proof which is based on Theorem.

Fix $a\in\hat{\mathbf{C}}-E(\{f^k\})$, which equals $\hat{\mathbf{C}}-E_V(\{f^k\})$ by Theorem. Let μ_f be any limit point of the sequence of $\{(f^k)^*\delta_a/d^k\}$. For every smooth function ϕ on $\hat{\mathbf{C}}$,

$$\left|\int \phi \mathrm{d}\left(\frac{(f^k)^*(\delta_a - \sigma)}{d^k}\right)\right| = \left|\frac{1}{d^k}\int \phi \mathrm{dd}^c\left(\log\frac{1}{[f^k, a]}\right)\right| \leq C_\phi \frac{1}{d^k}\int \log\frac{1}{[f^k, a]}\,\mathrm{d}\sigma, \tag{5}$$

where $C_\phi := \sup_{\hat{\mathbf{C}}}(|\mathrm{dd}^c\phi|/\sigma) < \infty$. Then (5) converges to 0 as $k \to \infty$, which in fact implies that $(f^k)^*\sigma/d^k \to \mu_f$ as $k \to \infty$ weakly. □

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