Derivatives of multiple sine functions

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(Communicated by Heisuke HIRONAKA, M. J. A., May 12, 2004)

Abstract: We calculate derivatives of multiple sine functions to investigate coefficients appearing in the addition type formula. We present explicit expressions and we obtain an interesting modular form.

Key words: Multiple sine function; multiple gamma function; Stirling modular form; zeta function.

1. Introduction.

Let

$$S_r(x,\underline{\omega}) = \prod_{n_1,\dots,n_r=0}^{\infty} (n_1\omega_1 + \cdots + n_r\omega_r + x) \times \left(\prod_{m_1,\dots,m_r=1}^{\infty} (m_1\omega_1 + \cdots + m_r\omega_r - x) \right)^{(-1)^{r-1}}$$

be the multiple sine function of period $\underline{\omega} = (\omega_1,\dots,\omega_r)$, where the zeta regularized product $\prod$ of Deninger [D] is used. Alternatively, $S_r(x,\underline{\omega})$ is written as

$$S_r(x,\underline{\omega}) = \Gamma_r(x,\underline{\omega})^{-1}\Gamma_r(\omega_1+\cdots+\omega_r - x,\underline{\omega})^{(-1)^r}$$

using the regularized multiple gamma function

$$\Gamma_r(x,\underline{\omega}) = \left(\prod_{n_1,\dots,n_r=0}^{\infty} (n_1\omega_1 + \cdots + n_r\omega_r + x) \right)^{-1} = \exp\left(\frac{\partial}{\partial s}\zeta_r(s,x,\underline{\omega})\Big|_{s=0} \right).$$

Here

$$\zeta_r(s,x,\underline{\omega}) = \sum_{n_1,\dots,n_r=0}^{\infty} (n_1\omega_1 + \cdots + n_r\omega_r + x)^{-s}$$

is the multiple Hurwitz zeta function defined by Barnes [B]. We denote by $\Gamma_r^B(x,\underline{\omega})$ the multiple gamma function studied in [B]:

$$\Gamma_r^B(x,\underline{\omega}) = \Gamma_r(x,\underline{\omega})\rho_r(\underline{\omega})$$

with

$$\rho_r(\underline{\omega}) = \lim_{x\to 0} \frac{\Gamma_r(x,\underline{\omega})^{-1}}{x}.$$

This $\rho_r(\underline{\omega})$ is called as "Stirling modular form" by Barnes [B]. The function $\Gamma_r^B(x,\underline{\omega})$ satisfies

$$\lim_{x\to 0} \frac{\Gamma_r^B(x,\underline{\omega})^{-1}}{x} = 1.$$

The case $r=1$ is reduced to the usual gamma function and sine function. In fact

$$\Gamma_1(x,\omega) = \frac{\Gamma(\frac{x}{\omega})}{\sqrt{2\pi}}\omega^{\frac{x}{\omega}-\frac{1}{2}}$$

and

$$S_1(x,\omega) = \frac{2\pi}{\Gamma\left(\frac{x}{\omega}\right)\Gamma\left(1-\frac{x}{\omega}\right)} = 2\sin\left(\frac{\pi x}{\omega}\right).$$

We notice that

$$\rho_1(\omega) = \sqrt{\frac{2\pi}{\omega}}$$

and

$$\Gamma_1^B(x,\omega) = \Gamma\left(\frac{x}{\omega}\right)\omega^{\frac{x}{\omega}-1}.$$

For simplicity we write

$$\Gamma_r(x) = \Gamma_r(x,(1,\dots,1)),$$
$$\Gamma_r^B(x) = \Gamma_r^B(x,(1,\dots,1)),$$
$$\rho_r = \rho_r(1,\dots,1)$$

and

$$S_r(x) = S_r(x,(1,\dots,1)).$$

Concerning multiple sine functions we refer to Shintani [S], Manin [M] and previous papers [K1, K2, K3, KK1, KK2, KOW, KW1, KW2].

We are quite interested in the derivatives $S_r^{(m)}(0,\underline{\omega})$ since they give the coefficients of

$$\Phi(u,v) = \sum_{m,n=0}^{\infty} c_{mn}(\underline{\omega})u^m v^n$$

satisfying the addition formula

2000 Mathematics Subject Classification. 11M06.

$$S_r(x+y,\underline{\omega}) = \Phi\left(S_r(x,\underline{\omega}), S_r(y,\underline{\omega})\right)$$

around $x = y = 0$. For example

$$\Phi(u,v) = u + v + c_{11}(\underline{\omega})uv + c_{12}(\underline{\omega})uv^2 + c_{21}(\underline{\omega})u^2v + (\text{degree} \geq 4)$$

with

$$c_{11}(\underline{\omega}) = \frac{S_r''(0,\underline{\omega})}{S_r'(0,\underline{\omega})^2}$$

and

$$c_{12}(\underline{\omega}) = c_{21}(\underline{\omega}) = \frac{S_r'''(0,\underline{\omega})S_r'(0,\underline{\omega}) - S_r''(0,\underline{\omega})^2}{2S_r'(0,\underline{\omega})^4}.$$

Note that $S_r(0,\underline{\omega}) = 0$ and $S_r'(0,\underline{\omega}) \neq 0$. These $c_{mn}(\omega_1,\dots,\omega_r)$ are interesting "Stirling modular functions".

In this paper we study $S_r'(0,\underline{\omega})$, $S_r''(0,\underline{\omega})$ and $c_{11}(\underline{\omega})$. We notice that

$$S_1'(0,\omega) = \frac{2\pi}{\omega} = \rho_1(\omega)^2,$$

$$S_1''(0,\omega) = 0$$

and

$$c_{11}(\omega) = 0.$$

The first non-trivial result was obtained by Shintani [S] (proof of Proposition 5):

$$S_2'(0,(\omega_1,\omega_2)) = \frac{2\pi}{\sqrt{\omega_1\omega_2}} = \rho_1(\omega_1)\rho_1(\omega_2).$$

See also Jimbo-Miwa [JM] using [K4]. We generalize this result as follows:

Theorem 1. *For $\underline{\omega} = (\omega_1,\dots,\omega_r)$ and $j = 1,\dots,r$, let*

$$P_j(\underline{\omega}) = \prod_{1\leq i_1<i_2<\cdots<i_j\leq r} \rho_j(\omega_{i_1},\omega_{i_2},\dots,\omega_{i_j}).$$

Then $S_r'(0,\underline{\omega})$ is expressed as follows:

(1) *If r is odd*

$$S_r'(0,\underline{\omega}) = \rho_r(\underline{\omega})^2 \prod_{j=1}^{r-1} P_j(\underline{\omega})^{(-1)^{j-1}}.$$

(2) *If r is even*

$$S_r'(0,\underline{\omega}) = \prod_{j=1}^{r-1} P_j(\underline{\omega})^{(-1)^{j-1}}.$$

Examples.

(1) $S_3'(0,(\omega_1,\omega_2,\omega_3))$

$$= \frac{\rho_3(\omega_1,\omega_2,\omega_3)^2\rho_1(\omega_1)\rho_1(\omega_2)\rho_1(\omega_3)}{\rho_2(\omega_1,\omega_2)\rho_2(\omega_2,\omega_3)\rho_2(\omega_3,\omega_1)}.$$

(2) $S_4'(0,(\omega_1,\omega_2,\omega_3,\omega_4)) =$

$$\frac{\rho_3(\omega_1,\omega_2,\omega_3)\rho_3(\omega_2,\omega_3,\omega_4)\rho_3(\omega_1,\omega_3,\omega_4)}{\rho_2(\omega_1,\omega_2)\rho_2(\omega_1,\omega_3)\rho_2(\omega_1,\omega_4)} \times \frac{\rho_3(\omega_1,\omega_2,\omega_4)\rho_1(\omega_1)\rho_1(\omega_2)\rho_1(\omega_3)\rho_1(\omega_4)}{\rho_2(\omega_2,\omega_3)\rho_2(\omega_2,\omega_4)\rho_2(\omega_3,\omega_4)}.$$

Since

$$\rho_1 = \sqrt{2\pi},$$

$$\rho_2 = \sqrt{2\pi}e^{-\zeta'(-1)}$$

and

$$\rho_3 = \sqrt{2\pi}e^{-\frac{1}{2}\zeta'(-2)-\frac{3}{2}\zeta'(-1)}$$

as calculated in [KK2], Theorem 1 gives concrete values:

Theorem 2.

(1) $$S_3'(0) = 2\pi e^{-\zeta'(-2)} = 2\pi\exp\left(\frac{\zeta(3)}{4\pi^2}\right).$$

(2) $$S_4'(0) = 2\pi e^{-2\zeta'(-2)} = 2\pi\exp\left(\frac{\zeta(3)}{2\pi^2}\right).$$

The calculation of $S_r''(0,\underline{\omega})$ is rather difficult and we report a few results below.

Theorem 3. *Let (ω_1,ω_2) satisfy $\omega_1 > 0$ and $\mathrm{Im}(\omega_2) > 0$. Put $\tau = \omega_2/\omega_1$, $q = e^{2\pi i\tau}$ and*

$$f(\tau) = \sum_{m,n=1}^{\infty} q^{mn} = \sum_{n=1}^{\infty} d(n)q^n.$$

Then we have the following result.

(1) $S_2''(0,(\omega_1,\omega_2))$

$$= \frac{8\pi^2 i}{\sqrt{\omega_1\omega_2^3}}\left\{\left(f\left(-\tfrac{1}{\tau}\right) - \frac{1}{4}\right) - \tau\left(f(\tau) - \frac{1}{4}\right)\right\}.$$

(2) $c_{11}(\omega_1,\omega_2)$

$$= \frac{2i}{\sqrt{\tau}}\left\{\left(f\left(-\tfrac{1}{\tau}\right) - \frac{1}{4}\right) - \tau\left(f(\tau) - \frac{1}{4}\right)\right\}.$$

Remark. For each complex number k we put

$$E_k(\tau) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$$

with

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$

Let

$$R_k(\tau) = E_k\left(-\tfrac{1}{\tau}\right) - \tau^k E_k(\tau).$$

Then

$$f(\tau) - \frac{1}{4} = E_1(\tau)$$

and

$$\left(f\left(-\tfrac{1}{\tau}\right)-\frac{1}{4}\right)-\tau\left(f(\tau)-\frac{1}{4}\right)=R_1(\tau).$$

Thus

$$\begin{aligned}c_{11}(\omega_1,\omega_2)&=\frac{2i}{\sqrt{\tau}}\left(E_1\left(-\tfrac{1}{\tau}\right)-\tau E_1(\tau)\right)\\&=\frac{2i}{\sqrt{\tau}}R_1(\tau).\end{aligned}$$

We notice that for an even integer $k \geq 4$, $E_k(\tau)$ is the Eisenstein series of weight k with respect to the modular group $SL_2(\mathbf{Z})$, so $R_k(\tau)=0$ in this case. For other k, $E_k(\tau)$ is a "fake Eisenstein series of weight k". For example

$$R_2(\tau)=-\frac{\tau^2}{4\pi i}.$$

It seems that $E_1(\tau)$ and $R_1(\tau)$ appear for the first time.

For $\underline{\omega}=(1,1)$ and $(1,1,1)$ we obtain concrete values.

Theorem 4.

(1) $S_2''(0)=-4\pi$.

(2) $c_{11}(1,1)=-\dfrac{1}{\pi}$.

(3) $S_3''(0)=-6\pi e^{-\zeta'(-2)}=-6\pi\exp\left(\dfrac{\zeta(3)}{4\pi^2}\right)$.

(4) $c_{11}(1,1,1)=-\dfrac{3e^{\zeta'(-2)}}{2\pi}=-\dfrac{3}{2\pi}\exp\left(-\dfrac{\zeta(3)}{4\pi^2}\right)$.

2. Proofs of theorems 1 and 2. We prove Theorem 1. Then Theorem 2 follows from Theorem 1 directly via the explicit values ρ_1, ρ_2 and ρ_3. We first show the case $r=2$ (Shintani's result). Recall that

$$S_2(x,(\omega_1,\omega_2))=\frac{\Gamma_2(\omega_1+\omega_2-x,(\omega_1,\omega_2))}{\Gamma_2(x,(\omega_1,\omega_2))}.$$

We use the basic periodicity

$$\Gamma_r(x+\omega_i,\underline{\omega})=\Gamma_r(x,\underline{\omega})\Gamma_{r-1}(x,\underline{\omega}(i))^{-1}$$

proved in [KK1], where

$$\underline{\omega}(i)=(\omega_1,\ldots,\omega_{i-1},\omega_{i+1},\ldots,\omega_r).$$

Then we have

$$\begin{aligned}&\Gamma_2(\omega_1+\omega_2-x,(\omega_1,\omega_2))\\&\quad=\Gamma_2(\omega_1-x,(\omega_1,\omega_2))\Gamma_1(\omega_1-x,\omega_1)^{-1}\\&\quad=\Gamma_2(-x,(\omega_1,\omega_2))\Gamma_1(-x,\omega_2)^{-1}\\&\qquad\qquad\times\Gamma_1(-x,\omega_1)^{-1}\Gamma_0(-x)\end{aligned}$$

with

$$\Gamma_0(x)=\frac{1}{x}.$$

Hence, by

$$\Gamma_r(x,\underline{\omega})^{-1}\sim\rho_r(\underline{\omega})x$$

we obtain

$$\begin{aligned}S_2'(0,(\omega_1,\omega_2))&=\lim_{x\to 0}\frac{S_2(x,(\omega_1,\omega_2))}{x}\\&=\frac{\rho_2(\omega_1,\omega_2)(-\rho_1(\omega_2))(-\rho_1(\omega_1))}{(-\rho_2(\omega_1,\omega_2))(-1)}\\&=\rho_1(\omega_1)\rho_1(\omega_2)\\&=\frac{2\pi}{\sqrt{\omega_1\omega_2}}.\end{aligned}$$

Hence we obtain the case $r=2$. The case of even r of Theorem 1 is similar to the case $r=2$ treated above. Hence it is sufficient to explain the case of odd r by looking at the following typical case $r=3$: from

$$\begin{aligned}&S_3(x,(\omega_1,\omega_2,\omega_3))\\&\quad=\Gamma_3(x,(\omega_1,\omega_2,\omega_3))^{-1}\\&\qquad\qquad\times\Gamma_3(\omega_1+\omega_2+\omega_3-x,(\omega_1,\omega_2,\omega_3))^{-1}\end{aligned}$$

and

$$\begin{aligned}&\Gamma_3(\omega_1+\omega_2+\omega_3-x,(\omega_1,\omega_2,\omega_3))\\&\quad=\Gamma_3(\omega_1+\omega_2-x,(\omega_1,\omega_2,\omega_3))\\&\qquad\qquad\times\Gamma_2(\omega_1+\omega_2-x,(\omega_1,\omega_2))^{-1}\\&\quad=\Gamma_3(\omega_1-x,(\omega_1,\omega_2,\omega_3))\Gamma_2(\omega_1-x,(\omega_1,\omega_3))^{-1}\\&\qquad\qquad\times\Gamma_2(\omega_1-x,(\omega_1,\omega_2))^{-1}\Gamma_1(\omega_1-x,\omega_1)\\&\quad=\Gamma_3(-x,(\omega_1,\omega_2,\omega_3))\Gamma_2(-x,(\omega_2,\omega_3))^{-1}\\&\qquad\qquad\times\Gamma_2(-x,(\omega_1,\omega_3))^{-1}\Gamma_1(-x,\omega_3)\\&\qquad\qquad\times\Gamma_2(-x,(\omega_1,\omega_2))^{-1}\Gamma_1(-x,\omega_2)\\&\qquad\qquad\times\Gamma_1(-x,\omega_1)\Gamma_0(-x)^{-1}\end{aligned}$$

by periodicity, we have

$$\begin{aligned}&S_3'(0,(\omega_1,\omega_2,\omega_3))\\&\qquad=\lim_{x\to 0}\frac{S_3(x,(\omega_1,\omega_2,\omega_3))}{x}\\&\qquad=\frac{\rho_3(\omega_1,\omega_2,\omega_3)^2\rho_1(\omega_1)\rho_1(\omega_2)\rho_1(\omega_3)}{\rho_2(\omega_1,\omega_2)\rho_2(\omega_2,\omega_3)\rho_2(\omega_3,\omega_1)}.\end{aligned}$$

The general case is easily obtained from

$$\begin{aligned}&S_r(x,(\omega_1,\ldots,\omega_r))\\&\quad=\Gamma_r(x,(\omega_1,\ldots,\omega_r))^{-1}\\&\qquad\qquad\times\Gamma_r(\omega_1+\cdots+\omega_r-x,(\omega_1,\ldots,\omega_r))^{(-1)^r}\end{aligned}$$

and

$$\begin{aligned}&\Gamma_r(\omega_1+\cdots+\omega_r-x,(\omega_1,\ldots,\omega_r))\\&\quad=\Gamma_r(\omega_1+\cdots+\omega_{r-1}-x,(\omega_1,\ldots,\omega_r))\\&\qquad\times\Gamma_{r-1}(\omega_1+\cdots+\omega_{r-1}-x,(\omega_1,\ldots,\omega_{r-1}))^{-1}\end{aligned}$$

via induction on r. □

3. Proof of theorem 3. From

$$S_2(x,(\omega_1,\omega_2))=S_2(x+\omega_2,(\omega_1,\omega_2))S_1(x,\omega_1)$$

we have

$$\begin{aligned}S_2'(x,(\omega_1,\omega_2))&=S_2'(x+\omega_2,(\omega_1,\omega_2))S_1(x,\omega_1)\\&\quad+S_2(x+\omega_2,(\omega_1,\omega_2))S_1'(x,\omega_1)\end{aligned}$$

and

$$\begin{aligned}S_2''(x,(\omega_1,\omega_2))&=S_2''(x+\omega_2,(\omega_1,\omega_2))S_1(x,\omega_1)\\&\quad+2S_2'(x+\omega_2,(\omega_1,\omega_2))S_1'(x,\omega_1)\\&\quad+S_2(x+\omega_2,(\omega_1,\omega_2))S_1''(x,\omega_1).\end{aligned}$$

Hence

$$S_2'(0,(\omega_1,\omega_2))=S_2(\omega_2,(\omega_1,\omega_2))S_1'(0,\omega_1)$$

and

$$S_2''(0,(\omega_1,\omega_2))=2S_2'(\omega_2,(\omega_1,\omega_2))S_1'(0,\omega_1).$$

In particular

$$\begin{aligned}S_2''(0,(\omega_1,\omega_2))&=2\frac{S_2'}{S_2}(\omega_2,(\omega_1,\omega_2))S_2'(0,(\omega_1,\omega_2))\\&=\frac{4\pi}{\sqrt{\omega_1\omega_2}}\cdot\frac{S_2'}{S_2}(\omega_2,(\omega_1,\omega_2)).\end{aligned}$$

Now we use the following expression due to Shintani [S] (Proposition 5):

$$S_2(x,(\omega_1,\omega_2))=e^{Q(x)}\frac{\prod_{n=0}^{\infty}\left(1-q^n\exp\left(\frac{2\pi ix}{\omega_1}\right)\right)}{\prod_{n=1}^{\infty}\left(1-q'^n\exp\left(\frac{2\pi ix}{\omega_2}\right)\right)}$$

with

$$q=e^{2\pi i\tau}=\exp\left(\frac{2\pi i\omega_2}{\omega_1}\right),$$

$$q'=e^{-2\pi i/\tau}=\exp\left(\frac{-2\pi i\omega_1}{\omega_2}\right)$$

and

$$\begin{aligned}Q(x)=\frac{\pi i}{2}\Bigg\{&\frac{x^2}{\omega_1\omega_2}-\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)x\\&+\frac{1}{6}\left(\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_2}\right)+\frac{1}{2}\Bigg\}.\end{aligned}$$

The logarithmic derivative gives

$$\begin{aligned}\frac{S_2'}{S_2}(x,(\omega_1,\omega_2))&=\frac{\pi i}{2}\left\{\frac{2x}{\omega_1\omega_2}-\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)\right\}\\&\quad-\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{2\pi i}{\omega_1}q^{nm}\exp\left(\frac{2\pi imx}{\omega_1}\right)\\&\quad+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{2\pi i}{\omega_2}q'^{nm}\exp\left(\frac{2\pi imx}{\omega_2}\right).\end{aligned}$$

Thus

$$\begin{aligned}&\frac{S_2'}{S_2}(\omega_2,(\omega_1,\omega_2))\\&\quad=\frac{\pi i}{2}\left(\frac{1}{\omega_1}-\frac{1}{\omega_2}\right)+\frac{2\pi i}{\omega_2}\left\{f\left(-\tfrac{1}{\tau}\right)-\tau f(\tau)\right\}\\&\quad=\frac{2\pi i}{\omega_2}\left\{\left(f(-\tfrac{1}{\tau})-\frac{1}{4}\right)-\tau\left(f(\tau)-\frac{1}{4}\right)\right\}.\end{aligned}$$

Hence we obtain $S_2''(0,(\omega_1,\omega_2))$ and $c_{11}(\omega_1,\omega_2)$. □

4. Proof of theorem 4. First recall the periodicity

$$S_2(x+1)=S_2(x)S_1(x)^{-1},$$
$$S_3(x+1)=S_3(x)S_2(x)^{-1}$$

and the differential equation

$$S_2'(x)=-S_2(x)\pi(x-1)\cot(\pi x),$$
$$S_3'(x)=S_3(x)\pi\frac{(x-1)(x-2)}{2}\cot(\pi x)$$

proved in [KK1, KK2].

(1) From

$$S_2(x)=S_2(x+1)S_1(x)$$

we have

$$S_2'(x)=S_2'(x+1)S_1(x)+S_2(x+1)S_1'(x)$$

and

$$\begin{aligned}S_2''(x)&=2S_2'(x+1)S_1'(x)+S_2''(x+1)S_1(x)\\&\quad+S_2(x+1)S_1''(x).\end{aligned}$$

Hence we obtain

$$S_2''(0)=4\pi S_2'(1)$$

from $S_1'(0)=2\pi$ and $S_1(0)=S_1''(0)=0$. By the way

$$\begin{aligned}S_2'(1)&=\lim_{x\to1}S_2'(x)\\&=-\lim_{x\to1}S_2(x)\cdot\frac{\pi(x-1)}{\tan\pi(x-1)}\\&=-S_2(1)\\&=-1\end{aligned}$$

since $S_2(1)=1$. Thus $S_2''(0)=-4\pi$.

(2) We have

$$c_{11}(1,1) = \frac{S_2''(0)}{S_2'(0)^2} = \frac{-4\pi}{(2\pi)^2} = -\frac{1}{\pi}$$

since $S_2''(0) = -4\pi$ by (1) and $S_2'(0) = 2\pi$ from Shintani's result.

(3) From

$$S_3(x) = S_3(x+1)S_2(x)$$

we have

$$S_3''(x) = S_3''(x+1)S_2(x) + 2S_3'(x+1)S_2'(x) + S_3(x+1)S_2''(x),$$

hence

$$S_3''(0) = 4\pi S_3'(1) - 4\pi S_3(1)$$

by $S_2(0) = 0, S_2'(0) = 2\pi$ and $S_2''(0) = -4\pi$. On the other hand we proved that

$$S_3(1) = e^{-\zeta'(-2)} = \exp\left(\frac{\zeta(3)}{4\pi^2}\right)$$

in [KK1]. Now we calculate $S_3'(1)$ as

$$\begin{aligned} S_3'(1) &= \lim_{x\to 1} S_3'(x) \\ &= \lim_{x\to 1} S_3(x)\pi\frac{(x-1)(x-2)}{2}\cot(\pi x) \\ &= \lim_{x\to 1} S_3(x)\frac{\pi(x-1)}{\tan\pi(x-1)}\cdot\frac{x-2}{2} \\ &= -\frac{1}{2}S_3(1). \end{aligned}$$

Hence

$$\begin{aligned} S_3''(0) &= -6\pi S_3(1) \\ &= -6\pi e^{-\zeta'(-2)} \\ &= -6\pi\exp\left(\frac{\zeta(3)}{4\pi^2}\right). \end{aligned}$$

(4) The above (3) and Theorem 2 (1) gives the result. □

Remark. By similar calculations on $S_2(x)$ we can show $c_{mn}(1,1) \in \mathbf{Q}(\pi)$ and $\Phi(u,v) \in \mathbf{Q}(\pi)[[u,v]]$, for example.

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