A class of balanced manifolds

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Abstract: We prove that a compact complex 3-dimensional manifold, which is Kähler outside a smooth curve, carries a balanced hermitian metric.

Key words: Kähler and balanced manifolds; Kähler currents.

1. Introduction. In this short note we would like to complete the analysis initiated in [2] and [3], showing that:

Corollary 2.9. Let $M$ be a compact complex manifold of dimension 3; if $M$ is Kähler outside a smooth curve $C$, then $M$ carries a balanced metric.

Recall that a hermitian metric on an n-dimensional manifold $M$ is called balanced, or $(p, q)$-forms (also called $(p, q)$-currents). A subscript $R$, for instance $D^p_{R}(M)$, denotes the spaces of real forms or currents.

We shall need the $\partial \overline{\partial}$-cohomology groups of $M$, or Aeppli groups, in particular for $p =1$ or $p = n-1$:

$$\Lambda^p_{R}(M) := \frac{\{\varphi \in D^p_{R}(M) : d\varphi = 0\}}{\{T \in \mathcal{D}^p_{R}(M) : dT = 0\}}.$$ 

Let $C$ be an irreducible curve in $M$. $C$ is the $(1, 1)$-component of a boundary means that the class of $[C]$, the current given by the integration on $C$, is the zero class in $\Lambda^{n-1, n-1}_{R}(M)$, while $C$ is part of the $(1, 1)$-component of a boundary means that there is a closed positive current $S \neq 0$ on $M$, of bidimension $(1, 1)$, such that $\chi C S = 0$ and the class of $[C] + S$ vanishes in $\Lambda^{n-1, n-1}_{R}(M)$.

The main result we need is the following (see [2], Theorem 5.5):

Theorem 2.1. Let $C$ be an irreducible curve in $M$, dim $M \geq 3$, such that $M - C$ is Kähler. Then one and only one of the following cases may occur:

(i) $M$ is Kähler,

(ii) $C$ is the $(1, 1)$-component of a boundary,

(iii) $C$ is part of the $(1, 1)$-component of a boundary.

Let us state a couple of Lemmas.

Lemma 2.2. Assume $C$ is a smooth curve in $M$. Then the map $i^* : \Lambda^{1, 1}_{R}(M) \to \Lambda^{1, 1}_{R}(C)$ induced by the embedding $i : C \to M$ is surjective if and only if $C$ is not the $(1, 1)$-component of a boundary.

Proof. $C$ is Kähler, because it is a curve, so that $\Lambda^{1, 1}_{R}(C) \simeq H^2(C, \mathbb{R}) \simeq \mathbb{R}$. Since $\Lambda^{1, 1}_{R}(C)$ is one-dimensional, $i^*$ is either surjective or the zero map.

Let us denote by “.” the intersection between classes in the Aeppli groups of complementary degree: f.i., for every closed form $\varphi \in \mathcal{D}^{1, 1}(M)$,
\[
\{\varphi\} \{[C]\} = \varphi.C = \int_C i^*\varphi = \int_M \varphi \wedge \psi,
\]
where \(\psi\) is a smooth representative of the class of \([C]\) in \(\Lambda_R^{n-1,n-1}(M)\).

If \(C\) is the \((1,1)\)-component of a boundary and \(i^*\) is surjective, take a volume form \(\beta\) on \(C\) and let \(\beta = i^*\varphi\); then
\[
0 = \varphi.C = \int_C i^*\varphi = \text{vol}(C) > 0.
\]

On the contrary, if \(C\) is not the \((1,1)\)-component of a boundary, since the intersection is non-degenerate, there is a class \(\{\varphi\} \neq 0\) such that
\[
0 \neq \varphi.C = \int_C i^*\varphi
\]
thus \(i^*\) is not the zero map.

**Lemma 2.3** ([4], Proposition 3.3). Let \(Y\) be an analytic subset of the compact complex space \(X\), let \(\gamma\) be a smooth representative of a class \([\gamma]\) \(\in \Lambda_R^{1,1}(X)\) such that \([\gamma]\)/\(Y\) is a Kähler class on \(Y\). Then there exists a smooth representative \(\gamma' = \gamma + \sqrt{-1}\partial\bar{\partial}u\) which is strictly positive on a neighborhood \(U\) of \(Y\).

Let us go back to Theorem 2.1. In the first case, \(M\) is obviously balanced. As regards the second case, the second author proved in [3] the following result:

**Theorem 2.4.** Let \(C\) be a smooth curve in a compact complex manifold \(M\) of dimension 3, such that \(M - C\) is Kähler. If \(C\) is the \((1,1)\)-component of a boundary, then \(M\) is balanced.

We shall consider here the third case, that is:

(*) \(M\) is a compact complex manifold of dimension \(n \geq 3\), \(i : C \to M\) is a smooth curve in \(M\) such that \(M - C\) is Kähler and \(C\) is part of the \((1,1)\)-component of a boundary.

We would like to prove that, in this situation, \(M\) belongs to the class \(\mathcal{C}\) of Fujiki, that is, it is bimeromorphic to a Kähler manifold. We can reach our goal thanks to the characterization of the class \(\mathcal{C}\) given in [4].

**Definition 2.5** ([4] or [6]). A Kähler current on a compact complex manifold \(M\) is a closed positive \((1,1)\)-current \(T\) which satisfies \(T \geq \epsilon\omega\) for some \(\epsilon > 0\) and some fundamental form \(\omega\) of a hermitian metric on \(M\).

**Theorem 2.6** ([4] Theorem 3.4). A compact complex manifold \(M\) admits a Kähler current if and only if \(M \in \mathcal{C}\).

**Theorem 2.7.** Assume (*). Then \(M \in \mathcal{C}\).

**Proof.** Let \(\beta\) be a closed Kähler form on \(M\). By Lemma 2.2, there is a form \(\gamma\) on \(M\) such that \([\gamma]\) = \([i^*\gamma]\) \(\in \Lambda_R^{1,1}(C)\). By Lemma 2.3, there is a smooth function \(u\) on \(M\) such that \(\gamma' = \gamma + \sqrt{-1}\partial\bar{\partial}u\) is strictly positive on \(U\), where \(U\) is a suitable neighborhood of \(C\). Let \(\alpha\) be the Kähler form of a Kähler metric on \(M - C\). Since \(\text{codim}C > 1\), \(\alpha\) extends to a closed positive current on \(M\) (see [5]), namely the trivial extension \(\alpha^0\).

Let \(\omega\) be a fixed fundamental form of a hermitian metric on \(M\). We claim that:

i) \(\exists \epsilon > 0\) such that \(\epsilon\omega < \gamma'\) on \(U\),

ii) \(\exists K > 0\) such that \(K\alpha^0 > \epsilon\omega - \gamma'\) on the compact set \(M - U\), because \(\alpha^0\) is smooth there.

Thus \(K\alpha^0 + \gamma' > \epsilon\omega\) on \(M\).

Thanks to the following theorem, we get the result we stated in the introduction.

**Theorem 2.8** ([1] Theorem 3.4). Let \(M\) and \(\tilde{M}\) be compact complex manifolds, and \(f : \tilde{M} \to M\) a modification. Then \(\tilde{M}\) is balanced if and only if \(M\) is balanced. In particular, all \(M \in \mathcal{C}\) are balanced.

**Corollary 2.9.** Let \(M\) be a compact complex manifold of dimension 3; if \(M\) is Kähler outside a smooth curve \(C\), then \(M\) carries a balanced metric.

**References**


