# A class of balanced manifolds

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**Abstract:** We prove that a compact complex 3-dimensional manifold, which is Kähler outside a smooth curve, carries a balanced hermitian metric.

**Key words:** Kähler and balanced manifolds; Kähler currents.

**1. Introduction.** In this short note we would like to complete the analysis initiated in [2] and [3], showing that:

**Corollary 2.9.** Let $M$ be a compact complex manifold of dimension 3; if $M$ is Kähler outside a smooth curve $C$, then $M$ carries a balanced metric.

Recall that a hermitian metric on an $n$-dimensional manifold $M$ is called balanced, or $(n-1)$-Kähler, if the fundamental form $\omega$ of the metric satisfies $d\omega^{n-1} = 0$.

This result is a step in proving the following very general conjecture:

**Conjecture** (see [7]). *Let $M$ be a compact complex manifold of dimension $n \geq 3$, and let $Y$ be an analytic subset of $M$ of codimension bigger than one. If $M - Y$ is Kähler, then $M$ is balanced.*

**Example.** The conjecture fails for $n = 2$, or when $codim Y = 1$.

Consider the Hopf surface $H$, as a principal bundle over $\mathbf{CP}_1$. It is well-known that $H$ is not Kähler, hence not balanced ($n = 2$), nor in the class $\mathcal{C}$ of Fujiki. Nevertheless, if $T$ denotes the generic fibre (which is a torus), $H - T \simeq \mathbf{C} \times T$ is Kähler. In dimension bigger than two, if $M$ is a Kähler manifold, $H \times M$ is Kähler outside the hypersurface $T \times M$, but is not balanced (if it were, the projection $H \times M \to H$ would imply that $H$ is balanced).

**2. Results.** Let us fix a compact complex manifold $M$ of dimension $n \geq 3$. $\mathcal{D}^{p,q}(M)$ and $\mathcal{D}'_{p,q}(M)$ are respectively the space of $(p,q)$-forms on $M$ and the space of currents on $M$ with *bidimension* $(p,q)$ or *bidegree* $(n-p, n-q)$ (also called $(n-p, n-q)$-currents). A subscript $\mathbf{R}$, for instance $\mathcal{D}^{p,p}_{\mathbf{R}}(M)$, denotes the spaces of *real* forms or currents.

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We shall need the $\partial\overline{\partial}$-cohomology groups of $M$, or Aeppli groups, in particular for $p = 1$ or $p = n-1$:

$$\Lambda^{p,p}_{\mathbf{R}}(M) := \frac{\{\varphi \in \mathcal{D}^{p,p}_{\mathbf{R}}(M) : d\varphi = 0\}}{\{\sqrt{-1}\partial\overline{\partial}\psi : \psi \in \mathcal{D}^{p-1,p-1}_{\mathbf{R}}(M)\}} \simeq \frac{\{T \in \mathcal{D}'_{n-p,n-p}(M)_{\mathbf{R}} : dT = 0\}}{\{\sqrt{-1}\partial\overline{\partial}P : P \in \mathcal{D}'_{n-p+1,n-p+1}(M)_{\mathbf{R}}\}}.$$

Let $C$ be an irreducible curve in $M$. *$C$ is the $(1,1)$-component of a boundary* means that the class of $[C]$, the current given by the integration on $C$, is the zero class in $\Lambda^{n-1,n-1}_{\mathbf{R}}(M)$, while *$C$ is part of the $(1,1)$-component of a boundary* means that there is a closed positive current $S \neq 0$ on $M$, of bidimension $(1,1)$, such that $\chi_C S = 0$ and the class of $[C] + S$ vanishes in $\Lambda^{n-1,n-1}_{\mathbf{R}}(M)$.

The main result we need is the following (see [2], Theorem 5.5):

**Theorem 2.1.** *Let $C$ be an irreducible curve in $M$, $\dim M \geq 3$, such that $M - C$ is Kähler. Then one and only one of the following cases may occur*:

(i) *$M$ is Kähler,*

(ii) *$C$ is the $(1,1)$-component of a boundary,*

(iii) *$C$ is part of the $(1,1)$-component of a boundary.*

Let us state a couple of Lemmas.

**Lemma 2.2.** *Assume $C$ is a smooth curve in $M$. Then the map $i^* : \Lambda^{1,1}_{\mathbf{R}}(M) \to \Lambda^{1,1}_{\mathbf{R}}(C)$ induced by the embedding $i : C \to M$ is surjective if and only if $C$ is not the $(1,1)$-component of a boundary.*

*Proof.* $C$ is Kähler, beacuse it is a curve, so that $\Lambda^{1,1}_{\mathbf{R}}(C) \simeq H^2(C, \mathbf{R}) \simeq \mathbf{R}$. Since $\Lambda^{1,1}_{\mathbf{R}}(C)$ is one-dimensional, $i^*$ is either surjective or the zero map.

Let us denote by "." the intersection between classes in the Aeppli groups of complementary degree: f.i., for every closed form $\varphi \in \mathcal{D}^{1,1}(M)$,

$$\{\varphi\}.\{[C]\} = \varphi.C = \int_C i^*\varphi = \int_M \varphi \wedge \psi,$$

where $\psi$ is a smooth representative of the class of $[C]$ in $\Lambda_{\mathbf{R}}^{n-1,n-1}(M)$.

If $C$ is the $(1,1)$-component of a boundary and $i^*$ is surjective, take a volume form $\beta$ on $C$ and let $\beta = i^*\varphi$: then

$$0 = \varphi.C = \int_C i^*\varphi = \mathrm{vol}(C) > 0.$$

On the contrary, if $C$ is not the $(1,1)$-component of a boundary, since the intersection is non-degenerate, there is a class $\{\varphi\} \neq 0$ such that

$$0 \neq \varphi.C = \int_C i^*\varphi$$

thus $i^*$ is not the zero map. □

**Lemma 2.3** ([4], Proposition 3.3). *Let $Y$ be an analytic subset of the compact complex space $X$, let $\gamma$ be a smooth representative of a class $\{\gamma\} \in \Lambda_{\mathbf{R}}^{1,1}(X)$ such that $\{\gamma\}/_Y$ is a Kähler class on $Y$. Then there exists a smooth representative $\gamma' = \gamma + \sqrt{-1}\partial\overline{\partial}u$ which is strictly positive on a neighborhood $U$ of $Y$.*

Let us go back to Theorem 2.1. In the first case, $M$ is obviously balanced. As regards the second case, the second author proved in [3] the following result:

**Theorem 2.4.** *Let $C$ be a smooth curve in a compact complex manifold $M$ of dimension 3, such that $M - C$ is Kähler. If $C$ is the $(1,1)$-component of a boundary, then $M$ is balanced.*

We shall consider here the third case, that is:

$(*)$ $M$ is a compact complex manifold of dimension $n \geq 3$, $i : C \to M$ is a smooth curve in $M$ such that $M - C$ is Kähler and $C$ is part of the $(1,1)$-component of a boundary.

We would like to prove that, in this situation, $M$ belongs to the class $\mathcal{C}$ of Fujiki, that is, it is bimeromorphic to a Kähler manifold. We can reach our goal thanks to the characterization of the class $\mathcal{C}$ given in [4].

**Definition 2.5** ([4] or [6]). A Kähler current on a compact complex manifold $M$ is a closed positive $(1,1)$-current $T$ which satisfies $T \geq \varepsilon\omega$ for some $\varepsilon > 0$ and some fundamental form $\omega$ of a hermitian metric on $M$.

**Theorem 2.6** ([4] Theorem 3.4). *A compact complex manifold $M$ admits a Kähler current if and only if $M \in \mathcal{C}$.*

**Theorem 2.7.** *Assume $(*)$. Then $M \in \mathcal{C}$.*

*Proof.* Let $\beta$ be a closed Kähler form on $C$. By Lemma 2.2, there is a form $\gamma$ on $M$ such that $\{\beta\} = \{i^*\gamma\} \in \Lambda_{\mathbf{R}}^{1,1}(C)$. By Lemma 2.3, there is a smooth function $u$ on $M$ such that $\gamma' := \gamma + \sqrt{-1}\partial\overline{\partial}u$ is strictly positive on $\overline{U}$, where $U$ is a suitable neighborhood of $C$. Let $\alpha$ be the Kähler form of a Kähler metric on $M - C$. Since *codim*$C > 1$, $\alpha$ extends to a closed positive current on $M$ (see [5]), namely the trivial extension $\alpha^0$.

Let $\omega$ be a fixed fundamental form of a hermitian metric on $M$. We claim that:

i) $\exists\, \varepsilon > 0$ such that $\varepsilon\omega < \gamma'$ on $\overline{U}$,

ii) $\exists\, K > 0$ such that $K\alpha^0 > \varepsilon\omega - \gamma'$ on the compact set $M - U$, because $\alpha^0$ is smooth there.

Thus $K\alpha^0 + \gamma' > \varepsilon\omega$ on $M$. □

Thanks to the following theorem, we get the result we stated in the introduction.

**Theorem 2.8** ([1] Theorem 3.4). *Let $M$ and $\tilde{M}$ be compact complex manifolds, and $f : \tilde{M} \to M$ a modification. Then $\tilde{M}$ is balanced if and only if $M$ is balanced. In particular, all $M \in \mathcal{C}$ are balanced.*

**Corollary 2.9.** *Let $M$ be a compact complex manifold of dimension 3; if $M$ is Kähler outside a smooth curve $C$, then $M$ carries a balanced metric.*

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