Uniqueness theorems concerning a question of Gross

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Abstract: In this paper, we deal with the problem of uniqueness of meromorphic functions concerning one question of Gross, and obtain some results that are improvements of that of former authors. Moreover, the example shows that the result is sharp.

Key words: Shared-set; uniqueness; meromorphic function.

1. Introduction and main results. In this paper, the term "meromorphic" will always mean meromorphic in the complex plane C. We assume that the reader is familiar with the basic results and notations of Nevanlinna's value distribution theory (see [4] or [5]), such as $T(r,f)$, $N(r,f)$ and $m(r,f)$. Meanwhile, we need the following notations. Let $f(z)$ be a meromorphic function. We denote by $n_{1)}(r,f)$ the number of simple poles of f in $|z| \le r$, $N_{1)}(r,f)$ is defined in terms of $n_{1)}(r,f)$ in the usual way (see [17]). We further define

$$\delta_{1)}(\infty,f) = 1 - \limsup_{r\to\infty} \frac{N_{1)}(r,f)}{T(r,f)}.$$

By the definition of $N_{1)}(r,f)$, we have

$$\begin{aligned} N_{1)}(r,f) &\le \overline{N}(r,f) \le \frac{1}{2} N_{1)}(r,f) + \frac{1}{2} N(r,f) \\ &\le \frac{1}{2} N_{1)}(r,f) + \frac{1}{2} T(r,f). \end{aligned}$$

From this we obtain

$$\begin{aligned} (1) \qquad \frac{1}{2}\delta_{1)}(\infty,f) &\le \frac{1}{2}\delta_{1)}(\infty,f) + \frac{1}{2}\delta(\infty,f) \\ &\le \Theta(\infty,f) \le \delta_{1)}(\infty,f). \end{aligned}$$

Let S be a subset of distinct elements in $\widehat{C}$. Define

$$E(S,f) = \bigcup_{a\in S} \{z | f(z) - a = 0, \text{ counting multiplicity}\},$$

$$\overline{E}(S,f) = \bigcup_{a\in S} \{z | f(z) - a = 0, \text{ ignoring multiplicity}\}.$$

Let f and g be two nonconstant meromorphic functions. If $E(S,f) = E(S,g)$, we say f and g share the set S CM (counting multiplicity). If $\overline{E}(S,f) = \overline{E}(S,g)$, we say f and g share the set S IM (ignoring multiplicity). Especially, let $S = \{a\}$, where $a \in \widehat{C}$, we say f and g share the value a CM if $E(S,f) = E(S,g)$, and say f and g share the value a IM if $\overline{E}(S,f) = \overline{E}(S,g)$ (see [16]).

In [3] F. Gross proved that there exist three finite sets S_j $(j = 1,2,3)$ such that any two nonconstant entire functions f and g satisfying $E(S_j,f) = E(S_j,g)$ for $j = 1,2,3$ must be identical, and asked the following question (see [3, Question 6]):

Question A. Can one find two finite sets S_j $(j = 1,2)$ such that any two entire functions f and g satisfying $E(S_j,f) = E(S_j,g)$ $(j = 1,2)$ must be identical?

H. Yi seems to have been the first to draw the affirmative answer to the above question A completely (see [12]). Since then, many results have been obtained for this and related topics (see [1, 2, 6–11, 14] and [15]).

In [3], F. Gross asked: "If the answer to Question 6 is affirmative, it would be interesting to know how large both sets would have to be." It is natural to ask the following question:

Question B. What are the smallest cardinalities of S_1 and S_2 respectively, where S_1 and S_2 are two finite sets such that any two entire functions f and g satisfying $E(S_j,f) = E(S_j,g)$ for $j = 1,2$ must be identical?

In 1998, H. Yi proved the following theorems.

Theorem A (see [16, Theorem 4]). *Let $S_1 = \{0\}$ and $S_2 = \{w | w^n(w+a) - b = 0\}$, where $n\,(\ge 2)$ is an integer, a and b are two nonzero constants such that the algebraic equation $w^n(w+a) - b = 0$ has no multiple roots. If f and g are two entire functions*

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satisfying $E(S_j, f) = E(S_j, g)$ *for* $j = 1, 2$, *then* $f \equiv g$.

Theorem B (see [16, Theorem 2]). *Let* S_1 *and* S_2 *are two finite sets such that any two entire functions* f *and* g *satisfying* $E(S_j, f) = E(S_j, g)$ *for* $j = 1, 2$ *must be identical, then* $\max\{\#(S_1), \#(S_2)\} \geq 3$, *where* $\#(S)$ *denotes the cardinality of the set* S.

From Theorem A and Theorem B, we immediately obtain that the smallest cardinalities of S_1 and S_2 are 1 and 3 respectively, where S_1 and S_2 are two finite sets such that any two entire functions f and g satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$ must be identical. Obviously, Theorem A and Theorem B answer the above Question B.

Now it is natural to ask the following question:

Question 1. What can be said if f and g are two meromorphic functions satisfying $E(\{\infty\}, f) = E(\{\infty\}, g)$ in Theorem A?

In this paper, we prove the following theorems, which answer Question 1.

Theorem 1. *Let* $S_1 = \{0\}$, $S_2 = \{\infty\}$ *and* $S_3 = \{w | w^n(w + a) - b = 0\}$, *where* $n\,(\geq 3)$ *is an integer,* a *and* b *are two nonzero constants such that the algebraic equation* $w^n(w+a)-b = 0$ *has no multiple roots. If* f *and* g *are two meromorphic functions satisfying* $E(S_j, f) = E(S_j, g)$ *for* $j = 1, 2, 3$ *and* $\Theta(\infty, f) > 0$, *then* $f \equiv g$.

Remark 1. The assumption "$\Theta(\infty, f) > 0$" in Theorem 1 can be replaced by "$\delta_{1)}(\infty, f) > 0$".

Remark 2. From (1), we know that $\Theta(\infty, f) > 0$ if and only if $\delta_{1)}(\infty, f) > 0$. This shows that Theorem 1 and Remark 1 are equivalent to each other. The following example shows that Theorem 1 is sharp.

Example 1. Let

$$f(z) = -\frac{ae^z(e^{nz} - 1)}{e^{(n+1)z} - 1}, \quad g(z) = -\frac{a(e^{nz} - 1)}{e^{(n+1)z} - 1}.$$

It is easy to see that f and g satisfy $E(S_j, f) = E(S_j, g)$ for $j = 1, 2, 3$, and $\Theta(\infty, f) = 0$ and $\delta_{1)}(\infty, f) = 0$. However, $f \not\equiv g$. This shows that the assumption "$\Theta(\infty, f) > 0$" in Theorem 1 is best possible.

Theorem 2. *Let* $S_1 = \{0\}$, $S_2 = \{\infty\}$ *and* $S_3 = \{w | w^n(w + a) - b = 0\}$, *where* $n\,(\geq 2)$ *is an integer,* a *and* b *are two nonzero constants such that the algebraic equation* $w^n(w + a) - b = 0$ *has no multiple roots. If* f *and* g *are meromorphic functions satisfying* $E(S_j, f) = E(S_j, g)$ *for* $j = 1, 2, 3$ *and* $\Theta(\infty,\ f) > 1/2$, *then* $f \equiv g$.

Theorem 3. *Let* S_1, S_2 *and* S_3 *be defined as in Theorem* 2. *If* f *and* g *are meromorphic functions satisfying* $E(S_j, f) = E(S_j, g)$ *for* $j = 1, 2, 3$ *and* $\delta_{1)}(\infty, f) > 5/6$, *then* $f \equiv g$.

Remark 3. Obviously, Theorem 2 and Theorem 3 answer the Question A and Question B posed by Gross. In case that f is an entire function, we have $\Theta(\infty, f) = \delta_{1)}(\infty, f) = 1$, so both of Theorem 2 and Theorem 3 improve Theorem A.

Suppose that f has no simple poles, then $\delta_{1)}(\infty, f) = 1$. Therefore, as an application of Theorem 3, we obtain the following result.

Corollary 1. *Let* S_1, S_2 *and* S_3 *be defined as in Theorem* 2. *If* f *and* g *are meromorphic functions satisfying* $E(S_j, f) = E(S_j, g)$ *for* $j = 1, 2, 3$ *and* f *has no simple poles, then* $f \equiv g$.

Remark 4. In case that f is an entire function, f has no simple poles, so Corollary 1 improves Theorem A.

Moreover, as an immediate consequence of Corollary 1, we have

Corollary 2. *Let* S_1, S_2 *and* S_3 *be defined as in Theorem* 2. *For a positive integer* k, *if* f *and* g *are meromorphic functions satisfying* $E(S_j, f^{(k)}) = E(S_j, g^{(k)})$ *for* $j = 1, 2, 3$, *then* $f^{(k)} \equiv g^{(k)}$.

2. Some Lemmas. In this section, f and g are two nonconstant meromorphic functions, and $S_3 = \{w | w^n(w+a)-b = 0\}$, where $n\,(\geq 2)$ is an integer, a and b are two nonzero constants such that the algebraic equation $w^n(w+a)-b = 0$ has no multiple roots. We denote by

$$F = \frac{f^n(f + a)}{b}, \quad G = \frac{g^n(g + a)}{b}. \tag{2}$$

Obviously, if $E(S_3, f) = E(S_3, g)$ then F and G share 1 CM.

Lemma 1. *Suppose that* $\overline{E}(\{0\}, f) = \overline{E}(\{0\}, g)$ *and* $\Theta(\infty, f) > 0$. *If* $F \equiv G$, *where* F *and* G *are defined as* (2), *then* $f \equiv g$.

Proof. Suppose that $f \not\equiv g$. Since $F \equiv G$, we have

$$f^n(f + a) = g^n(g + a), \tag{3}$$

and hence, f and g share 0, ∞ CM. Thus, we may assume that

$$\frac{f}{g} = e^{\alpha}, \tag{4}$$

where α is an entire function. By $f \not\equiv g$, we obtain that $e^{\alpha} \not\equiv 1$. From (3) and (4) we deduce

$$(5) \qquad f = -\frac{ae^{\alpha}(e^{n\alpha}-1)}{e^{(n+1)\alpha}-1}, \quad g = -\frac{a(e^{n\alpha}-1)}{e^{(n+1)\alpha}-1}.$$

Now we distinguish the following two cases.

Case 1. If e^{α} is a constant, then it follows from (5) that f is also constant. This is a contradiction.

Case 2. If e^{α} is nonconstant, then we have from (5) that

$$T(r,f) = nT(r,e^{\alpha}) + S(r,f),$$
$$\overline{N}(r,f) = nT(r,e^{\alpha}) + S(r,f).$$

It follows that $\Theta(\infty,f) = 0$. This contradicts $\Theta(\infty,f) > 0$.

This completes the proof of Lemma 1. $\square$

Lemma 2. *Let S_j $(j = 1,2,3)$ be defined as in Theorem 2, and let F and G be defined as (2). If $\overline{E}(S_1,f) = \overline{E}(S_1,g)$, $E(S_j,f) = E(S_j,g)$ for $j = 2,3$ and $F \not\equiv G$, then*

$$(6) \qquad \overline{N}\Big(r,\frac{1}{f}\Big) = \overline{N}\Big(r,\frac{1}{g}\Big) = S(r,f) + S(r,g).$$

Proof. Set

$$(7) \qquad H_1 := \frac{F'}{F-1} - \frac{G'}{G-1}.$$

Since $E(S_j,f) = E(S_j,g)$ for $j = 2,3$, we have F and G share 1, ∞ CM, and hence, (7) implies that $N(r,H_1) = S(r,f) + S(r,g)$. Moreover, by a logarithmic derivative theorem, we get that $m(r,H_1) = S(r,f) + S(r,g)$, so

$$(8) \qquad T(r,H_1) = S(r,f) + S(r,g).$$

We discuss the following two cases.

Case 1. Suppose that $H_1 \equiv 0$. By integration, we have from (7)

$$(9) \qquad F - 1 = A(G-1),$$

where A is a nonzero constant. Since $F \not\equiv G$, we have $A \neq 1$. Noting that $\overline{E}(S_1,f) = \overline{E}(S_1,g)$, from (2) and (9) we obtain that 0 is a Picard exceptional value of f and g. Thus, (6) holds.

Case 2. Suppose that $H_1 \not\equiv 0$. Assume that z_0 is a zero-point of f, by $\overline{E}(S_1,f) = \overline{E}(S_1,g)$, we obtain that z_0 is also zero-point of H_1. Thus, from this and (8), we have

$$\overline{N}\Big(r,\frac{1}{f}\Big) = \overline{N}\Big(r,\frac{1}{g}\Big) \leq N\Big(r,\frac{1}{H_1}\Big) = S(r,f) + S(r,g),$$

which proves Lemma 2. $\square$

Lemma 3. *Under the condition of Lemma 2, we have*

$$(10) \qquad \overline{N}(r,f) = \overline{N}(r,g) \leq \frac{1}{n}(T(r,f) + T(r,g)) + S(r,f) + S(r,g),$$

and

$$(11) \qquad \overline{N}(r,f) = \overline{N}(r,g) \leq \frac{1}{5}(T(r,f) + T(r,g)) + \frac{3}{5}N_{1)}(r,f) + S(r,f) + S(r,g).$$

Proof. Set

$$(12) \qquad H_2 := \Big(\frac{F'}{F-1} - \frac{G'}{G-1}\Big) - \Big(\frac{F'}{F} - \frac{G'}{G}\Big),$$

then

$$H_2 = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.$$

It follows that

$$(13) \qquad N(r,H_2) \leq \overline{N}\Big(r,\frac{1}{f}\Big) + \overline{N}\Big(r,\frac{1}{f+a}\Big) + \overline{N}\Big(r,\frac{1}{g+a}\Big).$$

Therefore, by a logarithmic derivative theorem, (6) and (13), we get that

$$(14) \qquad T(r,H_2) \leq T(r,f) + T(r,g) + S(r,f) + S(r,g).$$

We discuss the following two cases.

Case 1. Suppose that $H_2 \equiv 0$. By integration, we have from (12)

$$(15) \qquad \frac{F-1}{F} = B\frac{G-1}{G},$$

where B is nonzero constant. Since $F \not\equiv G$, we have $B \neq 1$. Again by (15), we deduce that ∞ is a Picard exceptional value of f. Therefore, (10) and (11) hold.

Case 2. Suppose that $H_2 \not\equiv 0$. Assume that z_1 is a pole of f with multiplicity p, then an elementary calculation gives that z_1 is the zero of H_2 with multiplicity at least $(n+1)p-1$. From this and (14), we obtain

$$(16) \qquad (2n+1)\,\overline{N}(r,f) - (n+1)\,N_{1)}(r,f) \leq N\Big(r,\frac{1}{H_2}\Big) \leq T(r,f) + T(r,g) + S(r,f) + S(r,g).$$

Noting that

$$(2n+1)\,\overline{N}(r,f) - (n+1)\,N_{1)}(r,f) \geq n\,\overline{N}(r,f),$$

we obtain from (16) that (10) holds. Again by (16), we have

$$(2n+1)\,\overline{N}(r,f) \le T(r,f)+T(r,g) + (n+1)\,N_{1)}(r,f)+S(r,f)+S(r,g).$$

It follows from $n \ge 2$ that (11) holds. This completes the proof of Lemma 3. □

Finally, we need the following important lemma due to Yi (see [13]). We first introduce some notations.

Let $F(z)$ be a meromorphic function, we denote by $n_2(r,F)$ the number of poles of F in $|z| \le r$, where a simple pole is counted once and a multiple pole is counted two times, $N_2(r,F)$ is defined as the counting function of $n_2(r,F)$. Moreover, we denote by E any set with finite linear measure.

Lemma 4. *Let F and G be two nonconstant meromorphic functions such that F and G share* 1, ∞ *CM. If*

$$N_2\Big(r,\frac{1}{F}\Big)+N_2\Big(r,\frac{1}{G}\Big)+2\,\overline{N}(r,F) < \lambda\,T(r)+S(r), \tag{17}$$

where $\lambda < 1$, $T(r) = \max\{T(r,F), T(r,G)\}$ and $S(r) = o\{T(r)\}$ $(r \to \infty, r \notin E)$, then $F \equiv G$ or $F\,G \equiv 1$.

3. Proof of main results.

3.1. Proof of Theorem 1. We define F and G as (2), then F and G share 1 CM.

Suppose that $F \not\equiv G$. Lemma 2 implies that $\overline{N}(r,\ 1/f) = \overline{N}(r,\ 1/g) = S(r)$. Therefore, we have

$$\begin{aligned} N_2\Big(r,\frac{1}{F}\Big) &+ N_2\Big(r,\frac{1}{G}\Big)+2\,\overline{N}(r,F) \\ &\le N\Big(r,\frac{1}{f+a}\Big)+N\Big(r,\frac{1}{g+a}\Big)+2\,\overline{N}(r,f)+S(r). \end{aligned} \tag{18}$$

Set $T_1(r) := \max\{T(r,f),\,T(r,g)\}$, then we obtain from (2) that

$$T(r) = (n+1)T_1(r)+O(1), \tag{19}$$

where $T(r) = \max\{T(r,F), T(r,G)\}$. From (10), (18) and (19) we deduce that

$$\begin{aligned} N_2\Big(r,\frac{1}{F}\Big)+N_2\Big(r,\frac{1}{G}\Big)+2\,\overline{N}(r,F) \\ \le \frac{2+\frac{4}{n}}{n+1}\,T(r)+S(r). \end{aligned} \tag{20}$$

Since $n \ge 3$, we have $2+(4/n) < n+1$. Using Lemma 4, we have $F\,G \equiv 1$. From (2) we obtain

$$f^n\,(f+a)\,g^n\,(g+a) \equiv b^2,$$

which implies that 0, $-a$ and ∞ are all Picard exceptional values of f. This is a contradiction. And hence, we obtain that $F \equiv G$, and by Lemma 1, we have that $f \equiv g$.

This completes the proof of Theorem 1. □

3.2. Proof of Theorem 2. If $n \ge 3$, by Theorem 1, we have $f \equiv g$. Next we assume that $n = 2$. Proceeding as in the proof of Theorem 1, we have (18) and (19). From (18) and (19), we deduce that

$$\begin{aligned} N_2\Big(r,\frac{1}{F}\Big)+N_2\Big(r,\frac{1}{G}\Big)+2\,\overline{N}(r,F) \\ \le \frac{4-2\,\Theta(\infty,f)}{3}\,T(r)+S(r). \end{aligned} \tag{21}$$

Noting that $4-2\Theta(\infty,f) < 3$ when $\Theta(\infty,f) > 1/2$, and using Lemma 4, we also obtain the conclusion of Theorem 2. □

3.3. Proof of Theorem 3. If $n \ge 3$, by Theorem 1, we have $f \equiv g$. Next we assume that $n = 2$. Proceeding as in the proof of Theorem 1, we also have (18) and (19). From (11) and (18), we deduce that

$$\begin{aligned} N_2\Big(r,\frac{1}{F}\Big)+N_2\Big(r,\frac{1}{G}\Big)+2\,\overline{N}(r,F) \\ \le \frac{14}{5}\,T_1(r)+\frac{6}{5}\,N_{1)}(r,f)+S(r). \end{aligned} \tag{22}$$

From (19) and (22) we obtain

$$\begin{aligned} N_2\Big(r,\frac{1}{F}\Big)+N_2\Big(r,\frac{1}{G}\Big)+2\,\overline{N}(r,F) \\ \le \frac{4-\frac{6}{5}\,\delta_{1)}(\infty,f)}{3}\,T(r)+S(r). \end{aligned}$$

Noting that $\delta_{1)}(\infty,f) > 5/6$ and using Lemma 4 and a similar method to the above proof, we obtain the conclusion of Theorem 3. □

4. Concluding Remarks.

In fact, we can obtain the following result from Section 3.

Remark 5. The assumption "$E(S_1,f) = E(S_1,g)$" in Theorem 1 can be replaced by "$\overline{E}(S_1,f) = \overline{E}(S_1,g)$".

Similarly, in Remark 1, Theorem 2, Theorem 3 the assumption "$E(S_1,f) = E(S_1,g)$"can be replaced by "$\overline{E}(S_1,f) = \overline{E}(S_1,g)$".

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