Real spectrum of ring of definable functions

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Abstract: Consider an o-minimal expansion of the real field. We deal with the real spectrums of the ring $C^r_{\mathrm{df}}(M)$ of definable C^r functions on an affine definable C^r manifold M in the present paper. Here r denotes a nonnegative integer. We show that the natural map $\mathrm{Sper}(C^r_{\mathrm{df}}(M)) \to \mathrm{Spec}(C^r_{\mathrm{df}}(M))$ is a homeomorphism when the o-minimal structure is polynomially bounded. If the o-minimal structure is not polynomially bounded, it is not known whether the natural map $\mathrm{Sper}(C^r_{\mathrm{df}}(M)) \to \mathrm{Spec}(C^r_{\mathrm{df}}(M))$ is a homeomorphism or not. However, the natural map $\mathrm{Sper}(C^0_{\mathrm{df}}(M)) \to \mathrm{Spec}(C^0_{\mathrm{df}}(M))$ is bijective even in this case.

Key words: O-minimal; real spectrum; Artin-Lang property.

1. Introduction. An o-minimal structure was first introduced by L. van den Dries [vdD1] and developed by A. Pillay, C. Steinhorn and so on [KPS, PS]. See [vdD2] for the definition and the geometric theory of o-minimal structures. We fix an o-minimal expansion of the real field in the present paper. Let M be an affine definable C^r manifold, where r denotes a nonnegative integer. An *affine definable C^r manifold* is a C^r submanifold of a Euclidean space $\mathbf{R}^n$ which is simultaneously a definable subset of $\mathbf{R}^n$. The notation $C^r_{\mathrm{df}}(M)$ denotes the ring of all definable C^r functions on M in the present paper. We want to study the ring $C^r_{\mathrm{df}}(M)$ from the real algebraic point of view in the present paper. If the reader is not familiar with the basic theory of real algebra, see [ABR, BCR].

The real spectrum of excellent rings has strong properties as introduced in [ABR, BCR]. In addition, it is known that the real spectrum of some large rings like the ring of continuous functions, abstract semialgebraic functions or real analytic functions on a 1-dimensional paracompact real analytic manifold coincides with the Zariski spectrum of them [AB, GR, GJ]. What about the ring $C^r_{\mathrm{df}}(M)$? In the present paper, we show that the natural map Φ_r : $\mathrm{Sper}(C^r_{\mathrm{df}}(M)) \to \mathrm{Spec}(C^r_{\mathrm{df}}(M))$ defined by

$$\Phi_r(\alpha) = \mathrm{supp}(\alpha) := \{f \in C^r_{\mathrm{df}}(M); f, -f \in \alpha\}$$

is a homeomorphism when the o-minimal structure is polynomially bounded. See [M1] for the definition of polynomially bounded o-minimal structures. On the other hand, it is not known whether the map Φ_r is a homeomorphism or not when the o-minimal structure is not polynomially bounded. However, we can show the natural continuous mapping Φ_0 : $\mathrm{Sper}(C^0_{\mathrm{df}}(M)) \to \mathrm{Spec}(C^0_{\mathrm{df}}(M))$ is bijective. They are the main results of the present paper.

In the present paper, r denotes the nonnegative integer. We abbreviate the sets $\{x \in M; f(x) \geq 0\}$, $\{x \in M; f(x) \geq 0,\ g(x) \geq 0\}$ *et al.* to $\{f \geq 0\}$, $\{f \geq 0,\ g \geq 0\}$ *et al.* when the domain of functions M is clear in the context. The notations $f(\alpha) > 0$, $f(\alpha) = 0$ and $f(\alpha) \leq 0$ denote the conditions $f \in \alpha \setminus \mathrm{supp}(\alpha)$, $f \in \mathrm{supp}(\alpha)$ and $-f \in \alpha$, respectively.

2. Artin-Lang property for definable C^r functions. Consider an o-minimal expansion $\tilde{\mathbf{R}}$ of the real field. Let M be an affine definable C^r manifold or a closed definable set when $r = 0$. By the same proof of [vdDM, Proposition C.9, Theorem C.11], we can show the following lemmas. We omit the proofs.

Lemma 2.1. *Let $f, g : M \to \mathbf{R}$ be continuous definable functions which are of class C^r on $M \setminus g^{-1}(0)$ with $f^{-1}(0) \subset g^{-1}(0)$. Then there exist an odd increasing definable C^r function $\phi : \mathbf{R} \to \mathbf{R}$ and a definable C^r function $h : M \to \mathbf{R}$ such that ϕ is a bijection and r-flat at 0 and $\phi \circ g = h \cdot f$. Furthermore, if $\tilde{\mathbf{R}}$ is polynomially bounded, we can choose a polynomial function $x \mapsto x^n$ as ϕ for some odd $n \in \mathbf{N}$.*

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Lemma 2.2. *Let A be a closed definable set of M, then A is the zero set of a definable C^r function on M.*

As a corollary of the above two lemmas, we can show the following lemma.

Lemma 2.3. *Let f be a definable C^r function on M. Set $A := \{x \in M; f(x) \geq 0\}$. Then there exist definable C^r functions $g, h : M \to \mathbf{R}$ such that*

$$g|_A \equiv 0,\ h^{-1}(0) \subset f^{-1}(0) \ \text{ and } \ h^2(x)f(x)+g(x) \geq 0$$

for all $x \in M$.

Proof. First define a continuous definable function $F : M \to \mathbf{R}$ by

$$F(x) := \begin{cases} \sqrt{-f(x)} & \text{if } x \notin A \\ 0 & \text{if } x \in A. \end{cases}$$

Remark that F is of class C^r outside of $F^{-1}(0)$. There exists a definable C^r function $G : M \to \mathbf{R}$ with $G^{-1}(0) = A$ by Lemma 2.2. Hence, by Lemma 2.1, there exist a definable C^r function $h : M \to \mathbf{R}$ and an odd increasing definable bijection $\phi : \mathbf{R} \to \mathbf{R}$ of class C^r with $\phi \circ G = h \cdot F$. Set $g := (\phi \circ G)^2$, then $g^{-1}(0) = A$. In this setting, it is obvious that $h^{-1}(0) \subset f^{-1}(0)$ and $h^2 f + g = h^2 f + h^2 F^2 \geq 0$. □

Let V_M denote the lattice consisting of all closed definable subsets of M. Define $\mathfrak{C}_M$ as the family of all prime V_M-filters. Consider $\mathfrak{C}_M$ as a topological space as follows: A subset U of $\mathfrak{C}_M$ is an open basis if there exists a finite sequence $f_1, \ldots, f_k \in C^r_{\mathrm{df}}(M)$ such that $U = \{\mathcal{F} \in \mathfrak{C}_M; V \notin \mathcal{F}\}$, where $V := \bigcup_{i=1}^k \{x \in M; f_i(x) \leq 0\}$.

We define maps between the space of all proper ideals of $C^r_{\mathrm{df}}(M)$ and the space of all V_M-filters.

Proposition 2.4. *For an ideal I of $C^r_{\mathrm{df}}(M)$, the family $\mathcal{Z}(I)$ of definable closed subsets of M defined as follows is a V_M-filter.*

$$\mathcal{Z}(I) := \{f^{-1}(0); f \in I\}$$

Conversely, for a V_M-filter $\mathcal{F}$, the subset $\mathcal{I}(\mathcal{F})$ of $C^r_{\mathrm{df}}(M)$ defined as follows is an ideal.

$$\mathcal{I}(\mathcal{F}) := \{f \in C^r_{\mathrm{df}}(M); f^{-1}(0) \in \mathcal{F}\}.$$

Furthermore, if $\mathcal{F}$ is a prime filter, the ideal $\mathcal{I}(\mathcal{F})$ is prime and the induced map

$$\mathcal{I} : \mathfrak{C}_M \to \mathrm{Spec}(C^r_{\mathrm{df}}(M))$$

is continuous.

Proof. We first show the first statement. Let $A, B \in \mathcal{Z}(I)$, then $A = f^{-1}(0)$ and $B = g^{-1}(0)$ for some $f, g \in I$. $A \cap B = (f^2+g^2)^{-1}(0) \in \mathcal{Z}(I)$. If $C \in V_M$ and $A \subset C$, there exists a definable C^r function $h : M \to \mathbf{R}$ with $C = h^{-1}(0)$ by Lemma 2.2. Then $C = (h \cdot f)^{-1}(0) \in \mathcal{Z}(I)$. It is obvious that $\emptyset \notin \mathcal{Z}(I)$.

We next show the second statement. Let $f, g \in \mathcal{I}(\mathcal{F})$ and $h \in C^r_{\mathrm{df}}(M)$. Set $A = f^{-1}(0)$ and $B = g^{-1}(0)$, then $A \cap B \in \mathcal{F}$. Then $\mathcal{F} \ni A \cap B \subset (f + g)^{-1}(0) \in \mathcal{F}$ by the definition of a V_M-filter. Hence $f + g \in \mathcal{I}(\mathcal{F})$. The product $h \cdot f$ is an element of $\mathcal{I}(\mathcal{F})$ because $\mathcal{F} \ni A \subset (h \cdot f)^{-1}(0) \in \mathcal{F}$.

We show the last statement. Assume that $\mathcal{F}$ is a prime V_M-filter. Let $f, g \in C^r_{\mathrm{df}}(M)$ with $f \cdot g \in \mathcal{I}(\mathcal{F})$. Then $f^{-1}(0) \cup g^{-1}(0) \in \mathcal{F}$. Since $\mathcal{F}$ is prime, $f^{-1}(0) \in \mathcal{F}$ or $g^{-1}(0) \in \mathcal{F}$. Hence $f \in \mathcal{I}(\mathcal{F})$ or $g \in \mathcal{I}(\mathcal{F})$.

Let $f \in C^r_{\mathrm{df}}(M)$. Then

$$\begin{aligned} &\mathcal{I}^{-1}(\{p \in \mathrm{Spec}(C^r_{\mathrm{df}}(M)); f \in p\}) \\ &\quad = \{\mathcal{F} \in \mathfrak{C}_M; f^{-1}(0) \in \mathcal{F}\}. \end{aligned}$$

Hence $\mathcal{I}$ is a continuous map. □

It is obvious that $I \subset \mathcal{I}(\mathcal{Z}(I))$ for any ideal I of $C^r_{\mathrm{df}}(M)$. Hence there exists a one-to-one correspondence between the space of all V_M-ultrafilters and $\mathrm{Specmax}(C^r_{\mathrm{df}}(M))$.

Corollary 2.5. *A prime ideal of $C^r_{\mathrm{df}}(M)$ is contained in only one maximal ideal.*

Proof. Let p be a prime ideal of $C^r_{\mathrm{df}}(M)$. Let m_1 and m_2 be two distinct maximal ideals containing p. There exist two distinct V_M-ultrafilters $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $m_1 = \mathcal{I}(\mathcal{F}_1)$ and $m_2 = \mathcal{I}(\mathcal{F}_2)$ as above. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are ultrafilters, there exist closed definable subsets $A_1 \in \mathcal{F}_1$ and $A_2 \in \mathcal{F}_2$ with $A_1 \cap A_2 = \emptyset$. Choose large definable closed subsets V_1 and V_2 of M such that $A_1 \subset V_1$, $A_2 \subset V_2$, $M = V_1 \cup V_2$, $A_2 \cap V_1 = \emptyset$ and $A_1 \cap V_2 = \emptyset$. There exist definable C^r functions $f_1, f_2 : M \to \mathbf{R}$ such that $V_1 = f_1^{-1}(0)$ and $V_2 = f_2^{-1}(0)$ by Lemma 2.2. By the definition, $f_1 \cdot f_2 \equiv 0$, $m_1 \ni f_1 \notin m_2$ and $m_1 \not\ni f_2 \in m_2$. Since p is real, $f_1 \in p$ or $f_2 \in p$. This contradicts the assumption that $p \subset m_1 \cap m_2$. □

Lemma 2.6. *Let $\mathcal{F}$ be a prime V_M-filter. Set*

$$\alpha(\mathcal{F}) := \{f \in C^r_{\mathrm{df}}(M); f^{-1}([0, +\infty)) \in \mathcal{F}\}.$$

Then $\alpha(\mathcal{F})$ is a prime cone with $\mathrm{supp}(\alpha) = \mathcal{I}(\mathcal{F})$.

Proof. It is easy to show this lemma. Hence we omit the proof. □

Lemma 2.7. *Let f be a nonnegative definable C^r function on M and α be a prime cone of $C^r_{\mathrm{df}}(M)$ such that* $\mathrm{supp}(\alpha) = \mathcal{I}(\mathcal{F})$ *for some prime V_M-filter $\mathcal{F}$. Then $f \in \alpha$.*

Proof. We may assume that $f \notin \mathrm{supp}(\alpha)$. Define a continuous definable function $F : M \to \mathbf{R}$ by

$$F(x) := \begin{cases} \sqrt{f(x)} & \text{if } f(x) > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

There exists a definable C^r function $G : M \to \mathbf{R}$ with $G^{-1}(0) = F^{-1}(0)$ by Lemma 2.2. By Lemma 2.1, there exist a definable C^r function $h : M \to \mathbf{R}$ and an odd increasing definable bijection $\phi : \mathbf{R} \to \mathbf{R}$ of class C^r with $\phi \circ G = h \cdot F$. Hence $h^2 f = (hF)^2 = (\phi \circ G)^2 \in \alpha$. Since $h^{-1}(0) \subset f^{-1}(0)$, $h \notin \mathrm{supp}(\alpha)$, and therefore, $h^2 \in \alpha \setminus \mathrm{supp}(\alpha)$. Therefore, $f \in \alpha$. □

Proposition 2.8. *Let $\mathcal{F}$ be a prime V_M-filter, then there exists a unique prime cone α of $C^r_{\mathrm{df}}(M)$ with* $\mathrm{supp}(\alpha) = \mathcal{I}(\mathcal{F})$.

Proof. The existence of α follows from Lemma 2.6.

We next show the uniqueness of α. Let β be a prime cone of $C^r_{\mathrm{df}}(M)$ with $\mathrm{supp}(\beta) = \mathcal{I}(\mathcal{F})$.

We will show that $\beta \subset \alpha$. Choose an arbitrary $f \in \beta$. We may assume without loss of generality that $f \notin \mathcal{I}(\mathcal{F})$. We lead contradiction under the assumption that $f \notin \alpha$, namely, $\{f \geq 0\} \notin \mathcal{F}$. Since $\mathcal{F}$ is prime, $\{f \leq 0\} \in \mathcal{F}$. By Lemma 2.3, there exists $g, h \in C^r_{\mathrm{df}}(M)$ such that $g \in \mathcal{I}(\mathcal{F})$, $h \notin \mathcal{I}(\mathcal{F})$ and $h^2 f + g \leq 0$ on M. Since $g \in \mathrm{supp}(\alpha)$, $h^2 f + g \in \alpha$. On the other hand, by Lemma 2.7, $-(h^2 f + g) \in \alpha$. Therefore, $h^2 f + g \in \mathrm{supp}(\alpha)$. Since $f \notin \mathrm{supp}(\alpha)$, h must be contained in $\mathrm{supp}(\alpha)$. This contradicts the condition $h^{-1}(0) \subset f^{-1}(0)$.

We will show the opposite inclusion $\alpha \subset \beta$. Let $f \in \alpha$. By the definition, $\{f \geq 0\} \in \mathcal{F}$. We may assume that $f \notin \mathcal{I}(\mathcal{F})$. There exist $h \notin \mathrm{supp}(\beta)$ and $g \in \mathrm{supp}(\beta)$ with $h^2 f + g \in \beta$ by Lemma 2.3 and Lemma 2.7. Hence, $h^2 f \in \beta$. Since $h \notin \mathrm{supp}(\beta)$, $f \in \beta$. □

We consider the case when $\tilde{\mathbf{R}}$ is polynomially bounded in the rest of this section.

Lemma 2.9. *Assume that $\tilde{\mathbf{R}}$ is polynomially bounded. Let p be a prime ideal of $C^r_{\mathrm{df}}(M)$, then the equation $p = \mathcal{I}(\mathcal{Z}(p))$ holds true.*

Proof. Set $\mathcal{F} := \mathcal{Z}(p)$. We have only to show that $p = \mathcal{I}(\mathcal{F})$. It is obvious that $p \subset \mathcal{I}(\mathcal{F})$. We show the opposite inclusion.

Let $g \in \mathcal{I}(\mathcal{F})$, then $g^{-1}(0) \in \mathcal{F}$. By the definition of $\mathcal{F}$, there exists $f \in p$ with $f^{-1}(0) = g^{-1}(0)$. There exist $n \in \mathbf{N}$ and $h \in C^r_{\mathrm{df}}(M)$ with $g^n = hf$ by Lemma 2.1. Since p is prime, $g \in p$ □

Lemma 2.10. *Assume that $\tilde{\mathbf{R}}$ is polynomially bounded. Let p be a prime ideal of $C^r_{\mathrm{df}}(M)$, then $\mathcal{Z}(p)$ is a prime V_M-filter. Furthermore, the induced map*

$$\mathcal{Z} : \mathrm{Spec}(C^r_{\mathrm{df}}(M)) \to \mathfrak{C}_M$$

is continuous.

Proof. We show the first statement. Let $A, B \in V_M$ such that $A \cup B \in \mathcal{Z}(p)$. There exist $f, g \in C^r_{\mathrm{df}}(M)$ and $h \in p$ with $f^{-1}(0) = A$, $g^{-1}(0) = B$ and $h^{-1}(0) = A \cup B$ by Lemma 2.2. By Lemma 2.1, there exist $n \in \mathbf{N}$ and $u \in C^r_{\mathrm{df}}(M)$ with $(fg)^n = uh \in p$. Since p is prime, $f \in p$ or $g \in p$, that is to say, $A \in \mathcal{Z}(p)$ or $B \in \mathcal{Z}(p)$.

We next show the last statement. Let U be an open basis of $\mathfrak{C}_M$. There exists a finite sequence $f_1, \ldots, f_k \in C^r_{\mathrm{df}}(M)$ such that $U = \{\mathcal{F} \in \mathfrak{C}_M; V \notin \mathcal{F}\}$, where $V := \bigcup_{i=1}^k \{x \in M; f_i(x) \geq 0\}$. By Lemma 2.2, there exists a definable C^r function g on M with $g^{-1}(0) = V$. We have only to show the equation

$$\mathcal{Z}^{-1}(U) = \{p \in \mathrm{Spec}(C^r_{\mathrm{df}}(M)); g \notin p\}$$

to show the last statement of this lemma. Let p be a prime ideal of $C^r_{\mathrm{df}}(M)$. First assume that $g \in p$. Then $V = g^{-1}(0) \in \mathcal{Z}(p)$. Hence $\mathcal{Z}(p) \notin U$. We next assume that $\mathcal{Z}(p) \notin U$, namely, $V \in \mathcal{Z}(p)$. Then $g \in \mathcal{I}(\mathcal{Z}(p)) = p$ by Lemma 2.9. We have shown the above equation and that the map $\mathcal{Z}$ is continuous. □

Theorem 2.11. *Consider a polynomially bounded o-minimal expansion of the real field. Fix a nonnegative integer r. Let M be an affine definable C^r manifold or a closed definable set when $r = 0$. Then the natural continuous map*

$$\Phi_r : \mathrm{Sper}(C^r_{\mathrm{df}}(M)) \to \mathrm{Spec}(C^r_{\mathrm{df}}(M))$$

is a homeomorphism and its inverse map is $\alpha \circ \mathcal{Z}$.

Proof. Since α and $\mathcal{Z}$ are continuous maps, we have only to show that $\beta = \alpha(\mathcal{Z}(\mathrm{supp}(\beta)))$ and $\mathrm{supp}(\alpha(\mathcal{Z}(p))) = p$ for all prime cones β of $C^r_{\mathrm{df}}(M)$. However, this equation is obvious by Proposition 2.8 and Lemma 2.9. □

Corollary 2.12 (Artin-Lang Property for definable C^r functions)**.** *Consider a polynomially bounded o-minimal expansion of the real field. Fix a nonnegative integer r. Let M be an affine definable C^r manifold or a closed definable set when $r = 0$. Then the continuous map*

$$\alpha : \mathfrak{C}_M \to \mathrm{Sper}(C^r_{\mathrm{df}}(M))$$

is a homeomorphism.

Proof. The mapping $\mathcal{Z}$ is a homeomorphism by Proposition 2.4, Lemma 2.9. Hence this corollary is obvious by Theorem 2.11. □

3. Real spectrum of ring of continuous definable functions. We showed the one-to-one correspondence between $\mathfrak{C}_M$ and $\mathrm{Spec}(C^r_{\mathrm{df}}(M))$ when $\tilde{\mathbf{R}}$ is polynomially bounded. However, this correspondence does not hold true when $\tilde{\mathbf{R}}$ is not polynomially bounded. The following example reveals this fact.

Example 3.1. Let $\tilde{\mathbf{R}}$ be an o-minimal expansion of the real field which is not polynomially bounded. Remember that the exponential function $\exp : \mathbf{R} \to \mathbf{R}$ is definable in $\tilde{\mathbf{R}}$ by [M2]. Fix a nonnegative integer r. Let $e : \mathbf{R} \to \mathbf{R}$ be the definable C^∞ function defined by

$$e(x) := \begin{cases} \exp\left(\frac{1}{x}\right) & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \exp\left(-\frac{1}{x}\right) & \text{if } x > 0. \end{cases}$$

We define an ideal I of $C^r_{\mathrm{df}}(\mathbf{R})$ as follows: A definable C^r function $f : \mathbf{R} \to \mathbf{R}$ is contained in I if, for any $n \in \mathbf{N}$ and $C > 0$, there exists $t > 0$ such that $|f(x)| \leq C \cdot x^n$ for $0 < x < t$. It is easy to see that I is a prime ideal. By the definition, $e(x) \in I$ and $x \notin I$. We next define a prime ideal J of $C^r_{\mathrm{df}}(\mathbf{R})$ as follows: A definable C^r function $f : \mathbf{R} \to \mathbf{R}$ is contained in J if $f(0) = 0$. It is obvious that $I \neq J = \mathcal{I}(\mathcal{Z}(I))$ and $J = \mathcal{I}(\mathcal{Z}(J))$. Hence $\mathcal{Z}$ is not injective and $\mathcal{Z}$ is not surjective.

Lemma 3.2. *Consider an o-minimal expansion of the real field and let M be a definable closed set. Let $\mathcal{F}$ be a prime V_M-filter and f be a continuous definable function on M. Then the following condtions are equivalent.*

1. $\{x \in M; f(x) \geq 0\} \in \mathcal{F}$
2. *There exists $g \in C^0_{\mathrm{df}}(M)$ such that $f - g^2 \in \mathcal{I}(\mathcal{F})$.*

Proof. First assume that $\{f \geq 0\} \in \mathcal{F}$. Define continuous definable functions $g, h : M \to \mathbf{R}$ by

$$g(x) = \begin{cases} \sqrt{f(x)} & \text{if } f(x) > 0 \\ 0 & \text{elsewhere} \end{cases}$$

and

$$h(x) = \begin{cases} f(x) & \text{if } f(x) < 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Then $h \in \mathcal{I}(\mathcal{F})$ and $f - g^2 = h$.

Conversely assume that $h := f - g^2 \in \mathcal{I}(\mathcal{F})$. Set $A := h^{-1}(0) \in \mathcal{F}$. Then f is nonnegative on A by the assumption. Hence $A \subset \{f \geq 0\}$, and therefore: $\{f \geq 0\} \in \mathcal{F}$. □

Lemma 3.3. *Consider an o-minimal expansion of the real field and let M be a definable closed set. Let p be a proper ideal of $C^0_{\mathrm{df}}(M)$. We define a subfamily $\mathcal{F}(p)$ of V_M as follows: The empty set is not contained in $\mathcal{F}(p)$ by definition and a nonempty closed definable subset S of M is an element of $\mathcal{F}(p)$ if and only if the ideal*

$$I(S) = \{f \in C^0_{\mathrm{df}}(M); f(x) = 0 (\forall x \in S)\}$$

is contained in p.

Then the family $\mathcal{F}(p)$ is a V_M-filter. Furthermore, if p is prime, so is $\mathcal{F}(p)$.

Proof. We first show that $\mathcal{F}(p)$ is a V_M-filter. By the definition, $\emptyset \notin \mathcal{F}(p)$. Let $S \in \mathcal{F}$ and T be a closed definable subset of M containing S. Since $I(T) \subset I(S)$, $I(T) \subset p$, namely, $T \in \mathcal{F}(p)$.

Let $A, B \in \mathcal{F}(p)$. We will show that $A \cap B \in \mathcal{F}(p)$. We have only to show that a continuous definable function $f : M \to \mathbf{R}$ with $A \cap B \subset f^{-1}(0)$ is contained in p. Define the continuous definable function $G : A \cup B \to \mathbf{R}$ as follows:

$$G(x) = \begin{cases} 0 & x \in A \\ f(x) & x \in B \end{cases}.$$

There exists a continuous definable function $g : M \to \mathbf{R}$ with $g|_{A \cup B} \equiv G$ by [vdD2, Corollary 8.3.10]. By the definition of g, $g \in I(A)$ and it is also obvious that $f - g \in I(B)$. Since $I(A) \subset p$ and $I(B) \subset p$ by the definition, $f \in p$. We have shown that $I(A \cap B) \subset p$ and finished to show that $\mathcal{F}(p)$ is a V_M-filter.

We next show the last statement of this lemma. Let V and W be definable closed subsets of M such that $V \cup W \in \mathcal{F}(p)$. We lead the contradiction under the assumption that $V, W \notin \mathcal{F}(p)$. There exist definable continuous functions $u \in I(V) \setminus p$ and $v \in I(W) \setminus p$. Then the function $u \cdot v$ vanishes on $V \cup W$, hence, $u \cdot v \in I(V \cup W) \subset p$. Since p is a prime ideal, $u \in p$ or $v \in p$. Contradiction. □

Lemma 3.4. *Consider an o-minimal expansion of the real field and let M be a definable closed set. Let p be a prime ideal of $C^0_{\mathrm{df}}(M)$ and $\mathcal{F}(p)$ be* *the prime V_M-filter defined in Lemma* 3.3. *Let f be a continuous definable function on M such that*

$$\{x \in M; |f(x)| \le g(x)\} \in \mathcal{F}(p)$$

for some $g \in p$. Then $f \in p$.

Proof. We first reduce to the case when $\{|f| \le g\} = M$. Set $h(x) := \max(0, |f(x)| - g(x))$, then $h \in \mathcal{I}(\mathcal{F}(p)) \subset p$. Replace g with $g + h$, then the condition $\{|f| \le g\} = M$ holds true.

First consider the case when $g^{-1}(0) \in \mathcal{F}(p)$. Then $f^{-1}(0) \in \mathcal{F}(p)$ because $g^{-1}(0) \subset f^{-1}(0)$. Hence $f \in \mathcal{I}(\mathcal{F}(p)) \subset p$.

We next consider the case when $g^{-1}(0) \notin \mathcal{F}(p)$. There exists $h \in C^0_{\mathrm{df}}(M)$ with $h \notin p$ and $h^{-1}(0) = g^{-1}(0)$ by the definition of $\mathcal{F}(p)$. Define a definable function $\phi : M \to \mathbf{R}$ by

$$\phi(x) := \begin{cases} \dfrac{f(x) \cdot h(x)}{g(x)} & \text{if } g(x) \neq 0 \\ 0 & \text{if } g(x) = 0. \end{cases}$$

The definable function ϕ is continuous because the function f/g on $\{x \in M; g(x) \neq 0\}$ is bounded. Hence $hf = \phi g \in p$. Since $h \notin p$, $f \in p$. □

Theorem 3.5. *Consider an o-minimal expansion of the real field and let M be a closed definable set or an affine definable manifold. Then the natural continuous mapping*

$$\Phi_0 : \mathrm{Sper}(C^0_{\mathrm{df}}(M)) \to \mathrm{Spec}(C^0_{\mathrm{df}}(M))$$

is bijective.

Proof. We first reduce to the case when M is a closed definable set. Let M be an affine definable manifold. We may assume that M is bounded in $\mathbf{R}^n$. Set $T = \overline{M} \setminus M$. There exists a continuous definable function $v : \mathbf{R}^n \to \mathbf{R}$ with $v^{-1}(0) = T$. Identify M with the image of M under the map $(\mathrm{id}, 1/v) : M \to \mathbf{R}^n$, then we may assume that M is closed in $\mathbf{R}^n$.

We have only to show that, for any prime ideal p of $C^0_{\mathrm{df}}(M)$, there exists a unique prime cone β of $C^0_{\mathrm{df}}(M)$ with $\mathrm{supp}(\beta) = p$. Let $\mathcal{F}(p)$ denote the prime filter defined in Lemma 3.3. Set $\beta := p \cup \alpha(\mathcal{F}(p))$. We will show that β is a prime cone of $C^0_{\mathrm{df}}(M)$. It is obvious that $-1 \notin \beta$. It is also easy to see that $ab \in \beta$ if $a, b \in \beta$. Let $a, b \in C^0_{\mathrm{df}}(M)$. Assume that $ab \in \beta$ and $a \notin \beta$. If $ab \in p$, then $b \in p$ because p is a prime ideal. Hence $-b \in p \subset \beta$. If $ab \in \alpha(\mathcal{F}(p))$, then $-b \in \alpha(\mathcal{F}(p)) \subset \beta$ because $\alpha(\mathcal{F}(p))$ is a prime cone.

We next show that $a + b \in \beta$ if $a, b \in \beta$. The claim is obvious when $a, b \in \alpha(\mathcal{F}(p))$ or $a, b \in p$. Hence we may assume that $a \in \alpha(\mathcal{F}(p)) \setminus p$ and $b \in p \setminus \alpha(\mathcal{F}(p))$. We will show that $a + b \in \alpha(\mathcal{F}(p))$. Assume the contrary, namely, that $A = \{a + b \le 0\} \in \mathcal{F}(p)$. Set $B = \{a \ge 0\} \in \mathcal{F}(p)$. Since $A \cap B \subset \{|a| \le -b\}$, $\{|a| \le -b\} \in \mathcal{F}(p)$. By Lemma 3.4, $a \in p$. Contradiction. We have shown that β is a prime cone. It is obvious that $\mathrm{supp}(\beta) = p$.

Let β' be a prime cone of $C^0_{\mathrm{df}}(M)$ with $p = \mathrm{supp}(\beta')$. Then $\beta' = \beta$. We will show this fact. We have only to show that $f \in \beta'$ if and only if $\{f \ge 0\} \in \mathcal{F}(p)$ for any $f \notin p$. If $\{f \ge 0\} \in \mathcal{F}(p)$, then $f - g^2 \in \mathcal{I}(\mathcal{F}(p)) \subset p$ for some $g \in C^0_{\mathrm{df}}(M)$ by Lemma 3.2. Since $g^2 \in \beta'$ by the definition of prime cones, $f \in \beta'$. Assume conversely that $\{f \ge 0\} \notin \mathcal{F}(p)$. Since $\mathcal{F}(p)$ is prime, $\{-f \ge 0\} \in \mathcal{F}(p)$. We can show that $-f \in \beta'$ in the same way as above, using Lemma 3.2. Since $f \notin \mathrm{supp}(\beta')$, $f \notin \beta'$. □

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