Nonunivalent generalized Koebe function

By Shinji Yamashita

Department of Mathematics, Tokyo Metropolitan University, 1-1, Minami-Osawa, Hachioji, Tokyo 192-0397

(Communicated by Heisuke Hironaka, m. j. a., Jan. 14, 2003)

Abstract: The function $f_\alpha(z) = \bigl(\{(1+z)/(1-z)\}^\alpha - 1\bigr)/(2\alpha)$ with a complex constant $\alpha \neq 0$ is not univalent in the disk $U = \{|z| < 1\}$ if and only if α is not in the union A of the closed disks $\{|z+1| \leqslant 1\}$ and $\{|z-1| \leqslant 1\}$. By making use of a geometric quantity we can describe how f_α "continuously tends to be" univalent in the whole U as α tends to each boundary point of A from outside.

Key words: Univalency; non-Euclidean disk; Schwarzian derivative.

1. Introduction. For a nonzero complex constant α let us define

$$f_\alpha(z) = \frac{1}{2\alpha}\left\{\left(\frac{1+z}{1-z}\right)^\alpha - 1\right\}$$

for z in $U = \{|z| < 1\}$, where the branch of the logarithm is chosen so that $\log 1 = 0$ in

$$\left(\frac{1+z}{1-z}\right)^\alpha = \exp\left(\alpha \log \frac{1+z}{1-z}\right).$$

The specified case of f_α is the Koebe function $f_2(z) = z/(1-z)^2$. In particular, $f'_\alpha(z) \neq 0$ for all $z \in U$.

It is a classical result of E. Hille [H] that f_α is univalent in U if and only if $\alpha \neq 0$ is in the union A of the closed disks $\{|z+1| \leqslant 1\}$ and $\{|z-1| \leqslant 1\}$. Note that z is in A if and only if $|z|^2 \leqslant 2|\operatorname{Re} z|$, whereas z is on the boundary ∂A of A if and only if $|z|^2 = 2|\operatorname{Re} z|$. Let ρ_α be the maximum of r, $0 < r \leqslant 1$, such that f_α is univalent in the non-Euclidean disk $\Delta(z,r) = \{w : |w-z|/|1-\overline{z}w| < r\}$ for each $z \in U$. The set $\Delta(z,r)$ actually is the Euclidean disk with the Euclidean center $(1-r^2)z/(1-r^2|z|^2)$ and the Euclidean radius $r(1-|z|^2)/(1-r^2|z|^2)$. Such a $\rho_\alpha > 0$ for $\alpha \notin A$ does exist as will be clarified in

Theorem. *Suppose that $\alpha \notin A$. If $i\alpha$ is real, then*

$$\rho_\alpha = \sqrt{\lambda + 1 - \sqrt{\lambda^2 + 2\lambda}}, \tag{1.1}$$

where $\lambda = 2/\sinh^2(\pi/|\alpha|)$. If $i\alpha$ is not real, then

$$\rho_\alpha \geqslant \sqrt{\mu + 1 - \sqrt{\mu^2 + 2\mu}}, \tag{1.2}$$

where $\mu = 2\cot^2(\pi|\operatorname{Re}\alpha|/|\alpha|^2)$. If α itself is real, then the equality holds in (1.2).

A consequence is that if $\beta \in \partial A$ and if $\alpha \notin A$ with $|\alpha - \beta| \to 0$, then $\rho_\alpha \to 1$. Namely, f_α "continuously tends to be" univalent in the whole U. This is obvious for $\beta \neq 0$ by (1.2) because $\mu \to 0$. For each sequence $\alpha_n \notin A$ with $\alpha_n \to 0$, both (1.1) and (1.2) show that $\rho_{\alpha_n} \to 1$.

2. Proof of the theorem. For z in the half-plane $H = \{z; \operatorname{Re} z > 0\}$ the set $\Delta_H(z,\rho) = \{w; |w-z|/|w+\overline{z}| < \rho\}$, $0 < \rho < 1$, is the image of $\Delta(T^{-1}(z),\rho)$ by the mapping $T(\zeta) = (1+\zeta)/(1-\zeta)$, and $\Delta_H(z,\rho)$ has the Euclidean center $c(z) = (z + \rho^2\overline{z})/(1-\rho^2)$ and the Euclidean radius $r(z) = (2\rho \operatorname{Re} z)/(1-\rho^2)$. Hence $\sin\theta = r(z)/|c(z)|$ with $0 < \theta < \pi/2$ and 2θ is the opening angle of $\Delta_H(z,\rho)$ viewed from the origin. Consequently,

$$\sin^2\theta = \frac{4X\rho^2}{\rho^4 + 2(2X-1)\rho^2 + 1} \tag{2.1}$$

for $X = \cos^2(\arg z)$, $|\arg z| < \pi/2$.

The image $\mathscr{D}$ of $\Delta_H(z,\rho)$ by $\log\zeta$ is contained in the rectangular domain of width

$$\log\frac{|c(z)| + r(z)}{|c(z)| - r(z)} = \log\frac{1+\sin\theta}{1-\sin\theta}$$

and of height 2θ. The boundary of $\mathscr{D}$ touches the rectangle at exactly four points.

Suppose first that $i\alpha$ is real. Then $\zeta^\alpha = \exp(\alpha\log\zeta)$ is univalent in $\Delta_H(z,\rho)$ if and only if

$$|\alpha|\log\frac{1+\sin\theta}{1-\sin\theta} \leqslant 2\pi.$$

To obtain the maximum $\rho(z)$ of ρ one has only to solve the equation $\sin\theta = \tanh(\pi/|\alpha|)$ and $\rho = \rho(z)$

2000 Mathematics Subject Classification. Primary 30C55.

in (2.1). After a short labor one then has

$$\rho(z)^2 = \lambda X + 1 - \sqrt{\lambda^2 X^2 + 2\lambda X}.$$

The right-hand side function of X attains its minimum at $X = 1$, namely, if and only if z is on the real axis, so that

$$\rho_\alpha^2 = \min_{z \in U} \rho(z)^2 = \lambda + 1 - \sqrt{\lambda^2 + 2\lambda}.$$

In the case where $i\alpha$ is not real, the function ζ^α is univalent in $\Delta_H(z, \rho)$ if $\theta \leqslant \pi |\operatorname{Re}\alpha|/|\alpha|^2$ $(< \pi/2)$, or equivalently, if $\sin\theta \leqslant \delta$, where $\delta = \sin(\pi|\operatorname{Re}\alpha|/|\alpha|^2)$. Consequently, this time,

$$\rho(z)^2 \geqslant \mu X + 1 - \sqrt{\mu^2 X^2 + 2\mu X},$$

where $\rho(z)$ is again the maximum of ρ. Following the same lines as in the proof of (1.1), one finally observes (1.2). In particular, if α itself is real, then the function ζ^α is univalent in $\Delta_H(z, \rho)$ if and only if $\theta = \pi/|\alpha|$. It is now easy to prove that the equality holds in (1.2).

It is open to prove whether or not the equality holds in (1.2) for nonreal α.

It follows from (1.1) that $(1 - \rho_\alpha)e^{\pi/|\alpha|} \to 2$ as $\alpha \to 0$ along the imaginary axis B, whereas, it follows from (1.2) that

$$0 \leqslant \limsup \frac{1 - \rho_\alpha}{1 - 2|\operatorname{Re}\alpha|/|\alpha|^2} \leqslant \frac{\pi}{2}$$

as α tends to a point of ∂A within the complex plane minus A and B. In particular, if c is real, then

$$\lim_{|c| \to 2+0} \frac{1 - \rho_c}{|c| - 2} = \frac{\pi}{4}.$$

If α is not in A, one can prove that

$$\rho_\alpha \leqslant \sqrt{\frac{3}{|1 - \alpha^2|}}. \tag{2.2}$$

This is significant in case $|1 - \alpha^2| > 3$ or α is in the exterior of the specified Jordan curve, namely, the lemniscate $\Gamma = \{|1 - z^2| = 3\}$. In particular, it follows from (2.2) that $\rho_\alpha \to 0$ as $\alpha \to \infty$. Let us return to the Theorem for a moment. If $B \ni \alpha \to \infty$, then $(1 - e^{-2\pi/|\alpha|})\rho_\alpha \to 0$, whereas, if α is real and if $|\alpha| \to +\infty$, then $\rho_\alpha/|\alpha| \to 0$.

One can observe that A is contained in the interior of Γ except for 2 and -2, and ∂A touches Γ at 2 and -2 where both curves have the common tangents $\{\operatorname{Re} z = 2\}$ and $\{\operatorname{Re} z = -2\}$, respectively.

For the proof of (2.2) set $f = f_\alpha$ and $\rho = \rho_\alpha$ for simplicity, and further set

$$\|f\| \equiv \sup_{z \in u} (1 - |z|^2)^2 |S_f(z)| = 2|1 - \alpha^2|, \tag{2.3}$$

where $S_f = f'''/f' - (3/2)(f''/f')^2$ is the Schwarzian derivative of f. Fix $z \in U$ and set $T(w) = (\rho w + z)/(1 + \overline{z}\rho w)$, so that the function

$$f \circ T(w) = a_0 + a_1 w + a_2 w^2 + a_3 w^3 + \cdots$$

of w is univalent in U. It then follows from the Bieberbach theorem [B], [G, p. 35, Theorem 2] that

$$\begin{aligned} \rho^2 (1 - |z|^2)^2 |S_f(z)| &= |S_{f \circ T}(0)| \\ &= 6 \left| \frac{a_3}{a_1} - \left(\frac{a_2}{a_1} \right)^2 \right| \leqslant 6. \end{aligned}$$

Hence $\rho^2 \|f\| \leqslant 6$, so that (2.2) is immediate from this and (2.3).

References

[B] Bieberbach, L.: Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln. Preuss. Akad. Wiss. Sitzungsb. 940–955 (1916).

[G] Goodman, A. W.: Univalent Functions. Vol. I. Mariner Publ. Co., Tampa (1983).

[H] Hille, E.: Remarks on a paper by Zeev Nehari. Bull. Amer. Math. Soc., **55**, 552–553 (1949).