Trace formula of twisting operators of half-integral weight in the case of even conductors

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Abstract: Let $S(k+1/2, N, \chi)$ denote the space of cusp forms of weight $k+1/2$, level N, and character χ. Let R_ψ be a twisting operator for a quadratic primitive character ψ of even conductor and $\tilde{T}(n^2)$ the n^2-th Hecke operator. We give an explicit trace formula of $R_\psi \tilde{T}(n^2)$ on $S(k+1/2, N, \chi)$.

Key words: Trace formula; twisting operator; half-integral weight.

1. Introduction. Let k and N be positive integers with $4 \mid N$. Let χ be an even Dirichlet character defined modulo N with $\chi^2 = \mathbf{1}$. We denote the space of cusp forms of weight $k+1/2$, level N, and character χ by $S(k+1/2, N, \chi)$.

In the previous paper [U1], we calculated an explicit trace formula of twisting operator on $S(k+1/2, N, \chi)$ for a quadratic primitive character of odd conductor. The purpose of this paper is to report an explicit trace formula of twisting operator for a quadratic primitive character of even conductor. Details will appear in [U2].

2. Notation. The notation in this paper is the same as in the previous paper [U1]. So, see [U1] and [U3] for the details of notation. Here, we explain several notations for convenience.

Let $\mathrm{ord}_p(\cdot)$ be the additive valuation for a prime number p with $\mathrm{ord}_p(p) = 1$. Put $\mu := \mathrm{ord}_2(N)$ and $\nu = \nu_p := \mathrm{ord}_p(N)$ for any odd prime number p. Then we decompose $N = 2^\mu M$. Namely, M is the odd part of N.

3. Results. Let ψ be a quadratic primitive character with even conductor r. Then we can express the conductor r as follows:

$$r = 2^u L, \quad u = 2, 3,$$

and L is a squarefree positive odd integer.

We consider the following conditions.

$$(*2) \qquad L^2 \mid M \quad \text{and} \quad \begin{cases} \mu \geqq 5, & \text{if } \mathfrak{f}(\chi_2) = 8. \\ \mu \geqq 4, & \text{if } \mathfrak{f}(\chi_2) \mid 4. \end{cases}$$

$$(*3) \qquad L^2 \mid M \quad \text{and} \quad \mu \geqq 6.$$

Here, $\mathfrak{f}(\chi_2)$ is the conductor of the 2-primary component χ_2 of the character χ.

From now on, we impose the condition (*2) if $u = 2$ $\left(\Leftrightarrow \psi_2 = \left(\frac{-1}{\cdot}\right)\right)$ and the condition (*3) if $u = 3$ $\left(\Leftrightarrow \psi_2 = \left(\frac{\pm 2}{\cdot}\right)\right)$, where ψ_2 is the 2-primary component of ψ. From these conditions and the assumption $\psi^2 = \mathbf{1}$, we see that the twisting operator R_ψ of ψ:

$$f = \sum_{n \geqq 1} a(n) q^n \mapsto f \mid R_\psi := \sum_{n \geqq 1} a(n)\psi(n) q^n,$$
$$(q := \exp(2\pi\sqrt{-1}z), \ z \in \boldsymbol{C}, \ \mathrm{Im}\, z > 0)$$

fixes the space of cusp forms $S(k+1/2, N, \chi)$ (cf. [Sh, Lemma 3.6]).

In the case of $k = 1$, we need to make a certain modification. It is well-known that the space $S(3/2, N, \chi)$ contains a subspace $U(N; \chi)$ which corresponds to a space of Eisenstein series via Shimura correspondence and which is generated by theta series of special type (cf. [U1, §0(c)]). Let $V(N; \chi)$ be the orthogonal complement of $U(N; \chi)$ in $S(3/2, N, \chi)$. Then it is also well-known that $V(N; \chi)$ corresponds to a space of cusp forms via Shimura correspondence. Hence, we need to consider the subspace $V(N; \chi)$ in place of $S(3/2, N, \chi)$ in the case of $k = 1$. The subspaces $U(N; \chi)$ and $V(N; \chi)$ are fixed by the twisting operator R_ψ (See [U2] for a proof and refer also to [U1, p. 94]). Hence, R_ψ gives an operator also on the subspace $V(N; \chi)$.

Now we can state an explicit trace formula.

Theorem. *We use the same notation as above. Let* $\tilde{T}(n^2) = \tilde{T}_{k+1/2,N,\chi}(n^2)$ *be the* n^2*-th*

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Hecke operator for a positive integer n with $(n,N)=1$ (cf. [U1, §0(c)]). Then explicit trace formulas of $R_\psi \tilde{T}(n^2)$ on the spaces $S(k+1/2,N,\chi)$ (if $k \geqq 2$) and $V(N;\chi)$ (if $k=1$) are given as follows:

$$\operatorname{tr}\bigl(R_\psi \tilde{T}(n^2); S(k+1/2,N,\chi)\bigr) = t(p)+t(e)+t(h), \qquad (\textit{if } k \geqq 2).$$

$$\operatorname{tr}\bigl(R_\psi \tilde{T}(n^2); V(N,\chi)\bigr) = t(p)+t(e)+t(h)+t(d), \qquad (\textit{if } k=1).$$

Here, $t(p)$, $t(e)$, $t(h)$, $t(d)$ are the contributions from the parabolic, elliptic, hyperbolic, and degree part. They are given by the tables (1.1)–(1.4) below.

We explain general notation that we use in the tables below.

Let $\boldsymbol{Z}_+$ be the set of all positive integers. And for a real number x, let $[x]$ be the greatest integer less than or equal to x. We denote by $\left(\frac{\cdot}{\cdot}\right)$ the Kronecker symbol. See [M, p. 82] for a definition of this symbol.

For a prime number p, let $|\cdot|_p$ be the p-adic valuation with the normalization $|p|_p = p^{-1}$. Then put

$$L_2 := N \prod_{p|2L} |N|_p.$$

Let χ_p denote the p-primary component of χ for a prime divisor p of N. Furthermore we set $\chi_A := \prod_{p|A} \chi_p$ for an arbitrary divisor A of N.

(1.1) **Parabolic part:** $t(p)$.

We decompose $n = {n_0}^2 n_1$ (n_0, $n_1 \in \boldsymbol{Z}_+$, n_1: squarefree). For any $d \in \boldsymbol{Z}_+$, put

$$\delta_0(\sqrt{d}\,) := \begin{cases} 1, & \text{if } d \text{ is square.} \\ 0, & \text{otherwise.} \end{cases}$$

And let $\mathcal{O}(-d)$ be the order of discriminant $-d$ in the imaginary quadratic number field $\boldsymbol{Q}(\sqrt{-d})$, $h(-d)$ the number of proper ideal classes of the order $\mathcal{O}(-d)$, and $w(-d)$ a half of the number of units in $\mathcal{O}(-d)$. Then put $h'(-d) := h(-d)/w(-d)$.

$\boxed{\text{Case I. } (u=3)}$ $\quad \left(\Leftrightarrow \psi_2 = \left(\frac{\pm 2}{\cdot}\right)\right)$

Case I-1. $\underline{\mu = 6,7 \text{ and } \mathfrak{f}(\chi_2) = 8.}$ $\qquad t(p) = 0.$

Case I-2. $\underline{\mu \geqq 8}$, or $\underline{\mu = 6,7 \text{ and } \mathfrak{f}(\chi_2) \mid 4.}$

$$t(p) = 0, \qquad (\text{if } Ln \equiv \psi_2(-1) \pmod 4).$$

$$t(p) = (-1)^k\, \psi(-1)^k\, \chi(n)\, n^{k-1}\, 2^{[(\mu-3)/2]} \times \prod_{p|L} p^{[(\nu-1)/2]} \times \prod_{p|L_2} \left(p^{[\nu/2]} + \left(\frac{-8Ln}{p}\right)^{\nu} p^{[(\nu-1)/2]} \right) \times \sum_{0<a|n_0} h'(-8Ln/a^2),$$

$$(\text{if } Ln \equiv -\psi_2(-1) \pmod 4).$$

$\boxed{\text{Case II. } (u=2)}$ $\quad \left(\Leftrightarrow \psi_2 = \left(\frac{-1}{\cdot}\right)\right)$

Case II-1. $\underline{\mu \geqq 7}$, or $\underline{\mu = 5,6 \text{ and } \mathfrak{f}(\chi_2) \mid 4.}$

$$t(p) = (-1)^k\, \psi(-1)^k\, \chi(n)\, n^{k-1}\, 2^{[(\mu-2)/2]} \times \prod_{p|L} p^{[(\nu-1)/2]} \times \prod_{p|L_2} \left(p^{[\nu/2]} + \left(\frac{-Ln}{p}\right)^{\nu} p^{[(\nu-1)/2]} \right) \times \sum_{0<a|n_0} h'(-4Ln/a^2),$$

$$(\text{if } Ln \equiv 1 \pmod 4).$$

$$t(p) = 0, \qquad (\text{if } Ln \equiv 3 \pmod 4).$$

Case II-2. $\underline{\mu = 5,6 \text{ and } \mathfrak{f}(\chi_2) = 8.}$ $\qquad t(p) = 0.$

Case II-3. $\underline{\mu = 4.}$

$$t(p) = (-1)^k\, \psi(-1)^k\, \chi(n)\, n^{k-1} \prod_{p|L} p^{[(\nu-1)/2]} \times \prod_{p|L_2} \left(p^{[\nu/2]} + \left(\frac{-Ln}{p}\right)^{\nu} p^{[(\nu-1)/2]} \right) \times \sum_{0<a|n_0} h'(-4Ln/a^2)$$
$$- \frac{(-1)^k}{2}\, \chi_2(-1)\, n^{k-1/2}\, \delta_0(\sqrt{Ln}\,) \prod_{p|L} p^{[(\nu-1)/2]} \times \prod_{p|L_2} \left(p^{[\nu/2]} + p^{[(\nu-1)/2]}\right),$$

$$(\text{if } Ln \equiv 1 \pmod 4).$$

$$t(p) = (-1)^k\, \psi(-1)^k\, \chi_2(-1)\, \chi(n)\, n^{k-1} \times \prod_{p|L} p^{[(\nu-1)/2]} \times \prod_{p|L_2} \left(p^{[\nu/2]} + \left(\frac{-Ln}{p}\right)^{\nu} p^{[(\nu-1)/2]} \right) \times \sum_{0<a|n_0} h'(-4Ln/a^2),$$

$$(\text{if } Ln \equiv 3 \pmod 4).$$

(1.2) **Elliptic part:** $t(e)$.

$$t(e) = -\psi(-1)^k\, r^{1-k}\, \chi_L(-n) \prod_{p\mid L} p^{[(\nu-1)/2]} \times \sum_{\substack{0<s<2\sqrt{rn} \\ s:(*),\, s^2\equiv 0(2^{u+2})}} \pi_k(s,rn)\, h'(D)\, \alpha_D(m_1') \times 2^{\mathrm{ord}_2(m_1)}\left(1-\left(\frac{D}{2}\right)2^{-1}\right)c_2(s,0) \times \prod_{p\mid L_2}\left(p^{-\mathrm{ord}_p(s)} n_p(\theta_p)\right).$$

Here, the condition (*) of s is the following:

(*) $\quad \mathrm{ord}_p(s) \geqq \left[(\nu_p+1)/2\right]$ for all prime divisors p of L.

The other notation is defined as follows: We decompose $s^2-4rn = {m_1}^2 D$ with $m_1 \in \boldsymbol{Z}_+$ and a discriminant D of an imaginary quadratic field. We put $m_1' := m_1 \prod_{p\mid N} |m_1|_p$ and $\theta = \theta_p := \mathrm{ord}_p(sm_1)$ for any prime number p. Moreover we put a constant $\pi_k(s,rn) := (x^{2k-1}-y^{2k-1})/(x-y)$, where x, y are two roots of the equation $X^2-sX+rn=0$. For any positive integer A, we define a constant $\alpha_D(A)$ by

$$\alpha_D(A) := \prod_{q\mid A}\left\{(q^{e+1}-1)-\left(\frac{D}{q}\right)(q^e-1)\right\}/(q-1),$$

where $A=\prod_{q\mid A} q^e$ is the prime decomposition of A. The constant $h'(D)$ is the same as in the parabolic part $t(p)$. Finally, the constants $n_p(\theta_p)$ $(p \mid L_2)$ and $c_2(s,0)$ are given by the tables below.

Table of $n_p(\theta_p)$

Case (1) $\quad (p \mid L_2$ and $p \mid s)$

$$\chi_p(r)\chi_p(D)\times n_p(\theta_p) = \begin{cases} p^\theta\left(p^{[\nu/2]}+\left(\frac{D}{p}\right)^\nu p^{[(\nu-1)/2]}\right), & \text{if } \theta \geqq \left[(\nu+1)/2\right]. \\ \left(1+\left(\frac{D}{p}\right)\right)p^{2\theta}, & \text{if } \theta \leqq \left[(\nu-1)/2\right]. \end{cases}$$

Case (2) $\quad (p \mid L_2$, $p \nmid s$ and $p \mid D)$

$$\chi_p(r)\times n_p(\theta_p) = \begin{cases} \left\{(p^{[\nu/2]}+p^{[(\nu-1)/2]})p^{\theta+1}-(p^\nu+p^{\nu-1})\right\} \times (p-1)^{-1}, & \text{if } \theta \geqq \left[\nu/2\right]. \\ 0, & \text{if } \theta \leqq \left[\nu/2\right]-1. \end{cases}$$

Case (3) $\quad (p \mid L_2$, $p \nmid s$ and $p \nmid D)$

$$\chi_p(r)\times n_p(\theta_p) = \begin{cases} \left(p-\left(\frac{D}{p}\right)\right)\left(p^{[\nu/2]}+p^{[(\nu-1)/2]}\right)\left(p^\theta-p^{[\nu/2]}\right) \times (p-1)^{-1} \\ \quad +\left(p^{[\nu/2]}+\left(\frac{D}{p}\right)^\nu p^{[(\nu-1)/2]}\right)p^{[\nu/2]}, & \text{if } \theta \geqq \left[(\nu+1)/2\right]. \\ \left(1+\left(\frac{D}{p}\right)\right)p^{2\theta}, & \text{if } \theta \leqq \left[(\nu-1)/2\right]. \end{cases}$$

Table of $c_2(s,0)$

<u>Case I</u> $\quad \left(u=3 \Leftrightarrow \psi_2 = \left(\frac{\pm 2}{}\right)\right)$

In this case, it follows that $\mathrm{ord}_2(D)$ is odd and so $D' := D/4 \equiv 2 \pmod 4$ from the conditions on s and the assumption $u=3$. Then, the table of the case $D' = D/4 \equiv 2 \pmod 4$ is given as follows:

Case (I-1) $\quad (\mu \geqq 8)$

$$c_2(s,0) = \begin{cases} 0, & \text{if } \mathrm{ord}_2(s) < \mu/2. \\ 0, & \text{if } \mathrm{ord}_2(s) \geqq \mu/2,\ Ln \equiv \psi_2(-1) \pmod 4. \\ 2^{[(\mu-3)/2]}\chi_2(-n), & \text{if } \mathrm{ord}_2(s) \geqq \mu/2,\ Ln \equiv -\psi_2(-1) \pmod 4. \end{cases}$$

Case (I-2) $\quad (\mu = 6, 7)$

$$c_2(s,0) = \begin{cases} 0, & \text{if } \mathrm{ord}_2(s) < \mu/2. \\ 0, & \text{if } \mathrm{ord}_2(s) \geqq \mu/2,\ Ln \equiv \psi_2(-1) \pmod 4. \\ 2^{[(\mu-3)/2]}\chi_2(-n), & \text{if } \mathrm{ord}_2(s) \geqq \mu/2,\ Ln \equiv -\psi_2(-1) \pmod 4, \text{ and } \mathfrak{f}(\chi_2) \mid 4. \\ 0, & \text{if } \mathrm{ord}_2(s) \geqq \mu/2,\ Ln \equiv -\psi_2(-1) \pmod 4, \text{ and } \mathfrak{f}(\chi_2) = 8. \end{cases}$$

<u>Case II</u> $\quad \left(u=2 \Leftrightarrow \psi_2 = \left(\frac{-1}{}\right)\right)$

Case (II-1) $\quad (D \equiv 1 \pmod 4)$

$$c_2(s,0) = \begin{cases} 0, & \text{if } \mu \geqq 5. \\ \chi_2(-L), & \text{if } \mu = 4. \end{cases}$$

Case (II-2-1) $\quad (D' := D/4 \equiv 2 \pmod 4$ and $\mu \geqq 6)$

$$c_2(s,0) = 0.$$

Case (II-2-2) $(D' := D/4 \equiv 2 \pmod 4$ and $\mu = 5)$

$$c_2(s,0) = \begin{cases} 2\,\chi_2(-L) = 2\,\chi_2(n), \\ \quad \text{if} \quad Ln \equiv -1 \pmod 4 \text{ and } \mathfrak{f}(\chi_2) \mid 4. \\ 0, \quad \text{otherwise.} \end{cases}$$

Case (II-2-3) $(D' := D/4 \equiv 2 \pmod 4$ and $\mu = 4)$

$$c_2(s,0) = \chi_2(-L).$$

Case (II-3-1) $(D' := D/4 \equiv 3 \pmod 4$ and $\mu \geqq 8)$

$$c_2(s,0) = \begin{cases} 0, & \text{if } \operatorname{ord}_2(s) \leqq [\mu/2]-1. \\ 2^{[\mu/2]-1}\chi_2(-n), & \text{if } \operatorname{ord}_2(s) \geqq [\mu/2]. \end{cases}$$

Case (II-3-2) $(D' := D/4 \equiv 3 \pmod 4$ and $\mu = 7)$

$$c_2(s,0) = \begin{cases} 0, & \text{if } s \equiv 4 \pmod 8. \\ 4\chi_2(-n), & \text{if } s \equiv 0 \pmod 8 \text{ and } \mathfrak{f}(\chi_2) \mid 4. \\ -4\chi_2(-n), & \text{if } s \equiv 8 \pmod{16} \text{ and } \mathfrak{f}(\chi_2) = 8. \\ 4\chi_2(-n), & \text{if } s \equiv 0 \pmod{16} \text{ and } \mathfrak{f}(\chi_2) = 8. \end{cases}$$

Case (II-3-3) $(D' := D/4 \equiv 3 \pmod 4$ and $\mu = 5, 6)$

$$c_2(s,0) = \begin{cases} 0, & \text{if } s \equiv 4 \pmod 8. \\ 2^{[\mu/2]-1}\chi_2(-n), & \text{if } s \equiv 0 \pmod 8,\ \mathfrak{f}(\chi_2) \mid 4. \\ 0, & \text{if } s \equiv 0 \pmod 8,\ \mathfrak{f}(\chi_2) = 8. \end{cases}$$

Case (II-3-4) $(D' := D/4 \equiv 3 \pmod 4$ and $\mu = 4)$

$$c_2(s,0) = \chi_2(-L).$$

(1.3) **Hyperbolic part:** $t(h)$.

Case I. $u = 2$, $L = 1$ $\left(\Leftrightarrow \psi = \left(\frac{-1}{}\right)\right)$, and $\mu = 4$.

(In this case, we have $\mathfrak{f}(\chi_2) \mid 4$.)

$$t(h) = -(-1)^k\,\chi_2(-1) \times \sum_{\substack{s > 2\sqrt{n},\ s \equiv 0(2) \\ s^2-4n=\square}} \big((s-m)/2\big)^{2k-1} \prod_{p \mid M} m_p(\theta_p).$$

Here, $s^2 - 4n = \square$ means that $s^2 - 4n$ is a square integer. And we put $m := (s^2 - 4n)^{1/2}$. For any prime divisor p of M, we put $\theta = \theta_p := \operatorname{ord}_p(sm)$ and define a constant $m_p(\theta_p)$ by the following:

$$m_p(\theta_p) := \begin{cases} p^{[\nu/2]} + p^{[(\nu-1)/2]}, & \text{if } \theta \geqq \big[(\nu+1)/2\big]. \\ 2p^{\theta}, & \text{if } \theta \leqq \big[(\nu-1)/2\big]. \end{cases}$$

Case II. <u>All the other cases.</u> $\qquad t(h) = 0.$

(1.4) **Degree part:** $t(d)$.

Let $n = \prod_{p \mid n} p^{\tau}$ be the prime decomposition of n. The character χ is expressed as

$$\chi = \prod_{p \mid N} \left(\frac{p}{}\right)^{\alpha_p}, \qquad (\alpha_p = 0, 1),$$

because χ is a quadratic even character.

In these notations, we have

$$\begin{aligned} t(d) &= \psi(-1)\,\chi_2(\psi(-1))\,\chi_L(-n)\,\chi_{L_2}(r) \\ &\times \prod_{p \mid n} \frac{p^{\tau+1}-1}{p-1} \\ &\times \prod_{p \mid L_2} \left\{ \left[\frac{\nu_p - \alpha_p}{2}\right] + 1 + \left[\frac{\nu_p + \alpha_p - 1}{2}\right] \left(\frac{-rn}{p}\right) \right\} \\ &\times \begin{cases} 1, & \text{if } \mathfrak{f}(\chi_2) \mid 4 \text{ and } \mu \leqq 5. \\ 1 - \left(\frac{-1}{n}\right)\psi(-1), & \text{if } \mathfrak{f}(\chi_2) \mid 4 \text{ and } \mu \geqq 6. \\ 0, & \text{if } \mathfrak{f}(\chi_2) = 8 \text{ and } \mu \leqq 7. \\ \left(\frac{2}{n}\right)\left(1 - \left(\frac{-1}{n}\right)\psi(-1)\right), & \text{if } \mathfrak{f}(\chi_2) = 8 \text{ and } \mu \geqq 8. \end{cases} \end{aligned}$$

4. Final remark. Using the above explicit trace formula, we can obtain trace identities between the trace of $R_\psi \tilde{T}(n^2)$ and linear combinations of traces of Hecke operators of integral weight and Atkin-Lehner involutions. The details of these trace identities also will appear in [U2].

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