Dimension of the square of a compactum and local connectedness

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Abstract: We state that a locally $(n-1)$-connected compactum with integral cohomological dimension n has n-cohomological dimension modulo p for some prime p. As a consequence, the integral cohomological dimension of the square of such a space is $2n$. In particular, the dimension of the square of an n-dimensional, locally $(n-1)$-connected compactum is $2n$.

Key words: Dimension; cohomological dimension; locally connected.

1. Introduction. One of the most interesting topics in dimension theory is to estimate the dimension of the product of compacta. It is easy to give an answer about this in a class of complexes. However, as it is known well, the dimension of the product of two compacta, even if the square of a compactum, is not always the sum of the dimension of each factor (Pontryagin [P], Boltyanskii [Bol] and Borsuk [B3]).

Studying dimension theory in the class of Absolute Neighborhood Retract (abbreviated ANR)-spaces was inspired by Borsuk:

Theorem (Borsuk [B1]). *Let X be an n-dimensional ANR compactum. Then there exists a prime $p \in \mathcal{P}$ such that* $\text{c-dim}_{\mathbf{Z}/p} X = n$.

As it was noticed by Pontryagin, the cohomological dimension modulo p of the product of compacta is the sum of the ones of these compacta. Therefore the next follows.

Corollary (Borsuk [B3]). *Let X be an n-dimensional ANR compactum. Then the square of X has dimension $2n$.*

Based on facts above, the following problem was also proposed:

Problem (Borsuk [B3]). *Is it true that the equality* $\dim X \times Y = \dim X + \dim Y$ *holds for ANR's X and Y?*

On the grounds that every ANR has the homotopy type of a complex, the problem had been reasonable. In this direction, the first affirmative answer is by Kodama:

Theorem (Kodama [Ko]). *Let X be a 2-dimensional ANR compactum. Then X is dimension full-valued.*

Here recall that a compactum X is *dimension full-valued*, if

$$\dim X \times Y = \dim X + \dim Y$$

for every compactum Y.

Remark. We note that the theorem above is true for a 2-dimensional locally 1-connected compactum (cf. [D-Y]).

Unfortunately, we encounter difficulties over dimension 4, when we consider the dimension of the product space of two different ANR's (the problem in dimension 3 is still open).

Example (Dranishnikov [Dr]). For each prime $p \in \mathcal{P}$, there exist a 4-dimensional ANR compactum M_p such that $\dim M_p \times M_q \leq 7$ for $p \neq q$.

The next classical theorem followed from Bockstein theorems is seen to be of use in estimating the dimension of the square of a general compactum.

Theorem ([Bo1, Bo2]). *Let X be an n-dimensional compactum. Then the square of X has dimension $2n$, or $2n-1$.*

In this note, we examine the (cohomological) dimension of the square of a compactum with a good local connectivity, that is, our main theorem is:

Theorem. *Let X be a locally $(n-1)$-connected compactum having* $\text{c-dim}_{\mathbf{Z}} X = n$. *Then there exists a prime $p \in \mathcal{P}$ such that* $\text{c-dim}_{\mathbf{Z}/p} X = n$.

In particular, we have:

Corollary. *Let X be a locally $(n-1)$-connected compactum having* $\text{c-dim}_{\mathbf{Z}} X = n$. *Then* $\text{c-dim}_{\mathbf{Z}} X \times X = 2n$.

Corollary. *Let X be an n-dimensional locally $(n-1)$-connected compactum. Then* $\dim X \times X = 2n$.

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Here we note that an n-dimensional locally n-connected compactum is an ANR [B3]. Thus our results are an extension of Borsuk's theorem.

We finally recall that the (covering) dimension $\dim X$ of a compactum X is the smallest natural number n such that there exists an $(n+1)$-fold covering by arbitrarily fine open sets. The cohomological dimension $\text{c-dim}_G X$ of a compactum X with coefficients in an abelian group G is the largest integer n such that there exists a closed subset A of X with $H^n(X, A; G) \neq 0$, where $H^n(\ ; G)$ means the Čech cohomology with coefficients in G. Clearly, $\dim X \leq n$ implies that $\text{c-dim}_G X \leq n$ for all G.

Throughout this paper, $\mathbf{Z}$ is the additive group of all integers. $\mathbf{Z}_{(P)}$ is the ring of integers localized at a subset P of $\mathcal{P} = \{\text{all prime numbers}\}$. We denote by $\mathbf{Z}/p$ the cyclic group of order p.

For a brief historical view of cohomological dimension theory, we refer the reader to Dranishnikov [Dr]. For information of ANR's, see Borsuk [B3].

2. Dimension of the square. Recall that an inverse sequence $\mathcal{G} = (G_n, p_{n,n+1})$ of groups satisfies *the Mittag-Leffler condition* provided for each n there exists $k > n$ such that $\operatorname{Im} p_{n,k} = \operatorname{Im} p_{n,m}$ for all $m > k$. $\mathcal{G}$ is said to be *stable* if it is isomorphic to a group in the category of pro-groups.

Proposition. *Let X be a compactum having $\text{c-dim}_{\mathbf{Z}} X = n$ and $\check{H}^n(X; \mathbf{Z}) \neq 0$. If pro-$H_{n-1}(X; \mathbf{Z})$ is stable and pro-$H_n(X; \mathbf{Z})$ satisfies the Mittag-Leffler condition, then there exists a prime $p \in \mathcal{P}$ such that $\text{c-dim}_{\mathbf{Z}/p} X = n$.*

Proof. Without loss of generality, we may suppose that X is connected. It follows from a series of Dydak's theorems in [D1] that $\check{H}^n(X; \mathbf{Z}) / \operatorname{Tor} \check{H}^n(X; \mathbf{Z})$ is free and $\operatorname{Tor} \check{H}^n(X; \mathbf{Z})$ is finite. By assumption $\check{H}^n(X; \mathbf{Z}) \neq 0$, we have a prime $p \in \mathcal{P}$ such that $\check{H}^n(X; \mathbf{Z}/p) \neq 0$. Then it means $\text{c-dim}_{\mathbf{Z}/p} X = n$. □

Before showing our theorem, we introduce the term of a cohomology locally connectedness.

Definition. A compactum X is *cohomology locally n-connected with respect to* $\mathbf{Z}$, if for each $x \in X$ and neighborhood N of x, there exists a neighborhood M of x in N such that the inclusion-induced homomorphism

$$i^*_{\mathbf{Z}} \colon \widetilde{\check{H}}^k(N; \mathbf{Z}) \to \widetilde{\check{H}}^k(M; \mathbf{Z})$$

is trivial for $k \leq n$, where $\widetilde{\check{H}}^*()$ is the reduced Čech cohomology theory.

We need the relative version of Wilder's theorem by Dydak-Koyama.

Theorem (Dydak-Koyama [D-K, Theorem 2.2]). *Let X be cohomology locally n-connected with respect to $\mathbf{Z}$. If A_1, A_2, B_1 and B_2 are closed subsets of X with $(B_1, A_1) \subseteq (\operatorname{Int} B_2, \operatorname{Int} A_2)$, then the image of the inclusion-induced homomorphism*

$$i^*_{\mathbf{Z}} \colon \widetilde{\check{H}}^k(B_2, A_2; \mathbf{Z}) \to \widetilde{\check{H}}^k(B_1, A_1; \mathbf{Z})$$

is finitely generated for $k \leq n$.

Theorem. *Let X be a locally $(n-1)$-connected compactum having $\text{c-dim}_{\mathbf{Z}} X = n$. Then there exists a prime $p \in \mathcal{P}$ such that $\text{c-dim}_{\mathbf{Z}/p} X = n$.*

Proof. We may suppose that X is connected. Then we note from Kozlowski-Segal [Kz-S] and Dydak [D2, Theorem 3.2] that pro-$H_k(X; \mathbf{Z})$ is stable for $k \leq n-1$ and pro-$H_n(X; \mathbf{Z})$ satisfies the Mittag-Leffler condition.

If $\check{H}^n(X; \mathbf{Z}) \neq 0$, then we have a prime $p \in \mathcal{P}$ such that $\text{c-dim}_{\mathbf{Z}/p} X = n$ by Proposition.

Using an argument of Dydak-Koyama [D-K, Corollary 2.4], we shall show our theorem in case $\check{H}^n(X; \mathbf{Z}) = 0$. Then X must be cohomology locally n-connected with respect to $\mathbf{Z}$. Now, by Bockstein's theorem, there exists a prime $p \in \mathcal{P}$ such that $\text{c-dim}_{\mathbf{Z}_{(p)}} X = n = \text{c-dim}_{\mathbf{Z}} X$. Thus we have a closed subset A of X such that $\check{H}^n(X, A; \mathbf{Z}_{(p)}) \neq 0$. It follows from continuity of Čech cohomology that there exists a compact neighborhood B of A in X for which the inclusion induced homomorphism $i^*_{\mathbf{Z}_{(p)}} \colon \check{H}^n(X, B; \mathbf{Z}_{(p)}) \to \check{H}^n(X, A; \mathbf{Z}_{(p)})$ is non-trivial.

Here we note from the relative version of Wilder's theorem by Dydak-Koyama above that the image of $i^*_{\mathbf{Z}} \colon \check{H}^n(X, B; \mathbf{Z}) \to \check{H}^n(X, A; \mathbf{Z})$ is finitely generated. Thus, the next commutative diagram shows that $\check{H}^n(X, B; \mathbf{Z})$ has an element which is not divisible by p:

$$\begin{array}{ccc} \check{H}^n(X, B; \mathbf{Z}) \otimes \mathbf{Z}_{(p)} & \xrightarrow{\approx} & \check{H}^n(X, B; \mathbf{Z}_{(p)}) \\ {\scriptstyle i^*_{\mathbf{Z}} \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle i^*_{\mathbf{Z}_{(p)}}} \\ \operatorname{Im}(i^*_{\mathbf{Z}}) \otimes \mathbf{Z}_{(p)} & \longrightarrow & \check{H}^n(X, A; \mathbf{Z}_{(p)}) \end{array}$$

Therefore $\check{H}^n(X, B; \mathbf{Z}/p) \approx \check{H}^n(X, B; \mathbf{Z}) \otimes \mathbf{Z}/p \neq 0$, that is, $\text{c-dim}_{\mathbf{Z}/p} X = n = \text{c-dim}_{\mathbf{Z}} X$. □

Corollary. *Let X be a locally $(n-1)$-connected compactum having $\text{c-dim}_{\mathbf{Z}} X = n$. Then $\text{c-dim}_{\mathbf{Z}} X \times X = 2n$.*

Proof. By Theorem, we have a prime $p \in \mathcal{P}$ such that $\text{c-dim}_{\mathbf{Z}} X = \text{c-dim}_{\mathbf{Z}/p} X$. Thus,

$$2n \geq \text{c-dim}_{\mathbf{Z}} X \times X \geq \text{c-dim}_{\mathbf{Z}/p} X \times X$$
$$= \text{c-dim}_{\mathbf{Z}/p} X + \text{c-dim}_{\mathbf{Z}/p} X = 2n.$$

□

Corollary. *Let X be an n-dimensional locally $(n-1)$-connected compactum. Then* $\dim X \times X = 2n$.

Remark. Borsuk [B2] constructed a 2-dimensional locally connected compactum with $\dim X \times X = 3$. Therefore we cannot extend results above to more general cases.

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