Universality of Hecke L-functions in the Grossencharacter-aspect

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Abstract: We consider the Hecke L-function $L(s,\lambda^m)$ of the imaginary quadratic field $\mathbf{Q}(i)$ with the m-th Grossencharacter λ^m. We obtain the universality property of $L(s,\lambda^m)$ as both m and $t = \mathrm{Im}(s)$ go to infinity.

Key words: Hecke L-function; universality of zeta functions; Grossencharacter; imaginary quadratic field.

1. Introduction. Voronin [V] discovered the universality property of the Riemann zeta function in 1975, which is stated as follows:

Voronin's theorem. Let C be a compact subset of the strip $\{s = \sigma + i\tau \in \mathbf{C} \mid (1/2) < \sigma < 1\}$ with connected complement. Let $f(s)$ be a non-vanishing continuous function on C which is analytic in the interior of C. Then for any $\varepsilon > 0$,

$$\varliminf_{T\to\infty} \frac{\mu\Big(\Big\{t \in [0,T] \,\Big|\, \sup_{s\in C}|\zeta(s+it) - f(s)| < \varepsilon\Big\}\Big)}{T} > 0,$$

where μ is the Lebesgue measure on $\mathbf{R}$.

This result was extended to various zeta functions. The first author proved it for Hecke L-functions with ideal class characters [M1] and for those with Grossencharacters [M2]. The universality properties are also generalized to various aspects of zeta functions. Recently Nagoshi proved them for automorphic L-functions of $GL(2)$ in the aspect where their weight or level of the cusp forms grows [N1]. Nagoshi also generalized it to Maass cusp forms for $GL(2)$ in the aspect of the Laplace eigenvalues [N2].

In this paper we deal with the Hecke L-functions $L(s,\lambda^m)$ of $\mathbf{Q}(i)$ with Grossencharacters λ^m ($m \in \mathbf{Z}$), where λ is a fixed generator of Grossencharacters. We consider the universality property as both τ and m grow. More precisely our results are stated as follows:

Let $K = \mathbf{Q}(i)$, and for an ideal $\mathfrak{a} = (\alpha) \in K$, the m-th Grossencharacter is given by $\lambda^m(\mathfrak{a}) := (\alpha/|\alpha|)^{4m}$ for $m \in \mathbf{Z}$. The Hecke L-function is defined by $L(s,\lambda^m) = \sum_{\mathfrak{a}} \lambda^m(\mathfrak{a})N(\mathfrak{a})^{-s}$ for $\sigma = \mathrm{Re}(s) > 1$.

Theorem 1.1. *Let C be a compact subset in the strip $\{s \in \mathbf{C} \mid (1/2) < \sigma < 1\}$. For any function $f(s)$ which is nonzero and continuous on C and which is holomorphic on* $\mathrm{Int}(C)$, *and for any $\varepsilon > 0$, we have*

$$(1.1)\quad \varliminf_{T\to\infty} \frac{1}{T^2}\mu'\big(\{(t,m) \in [0,T] \times \{0,\dots,[T]\} \mid \max_{s\in C}|L(s+it,\lambda^m) - f(s)| < \varepsilon\}\big) > 0,$$

where μ' is the product measure on $\mathbf{R} \times \mathbf{Z}$.

Remark 1.2. (a) It is possible to extend Theorem 1.1 to any imaginary quadratic field K of class number one, and to general Hecke character $\chi\lambda^m$ with nontrivial narrow class character χ.

(b) In case K is a general number field of finite degree, (1.1) would be formulated as follows: Let $n = [K : \mathbf{Q}]$ and $\lambda_1,\dots,\lambda_{n-1}$ be a fixed set of generators of Grossencharacters of K. Put $\lambda^m = \lambda_1^{m_1}\cdots\lambda_{n-1}^{m_{n-1}}$ for $m = (m_1,\dots,m_{n-1}) \in \mathbf{Z}^{n-1}$. Then under the above settings we would have

$$\varliminf_{T\to\infty} \frac{1}{T^n}\mu'\Big(\Big\{(t,m) \in [0,T]^n \,\Big|\, \max_{s\in C}|L(s+it,\lambda^m) - f(s)| < \varepsilon\Big\}\Big) > 0$$

with μ' the product measure on $\mathbf{R}\times\mathbf{Z}^{n-1}$. This will be treated in the forthcoming paper [M3].

(c) In Theorem 1.1, it is unfortunate that the range of m and t must be the same. The universality in the m-aspect with t being fixed should

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also be proved. Difficulty lies in the proof of the mean value theorem for Dirichlet series over O_K twisted by λ^m. Duke proves it in [D, Theoreom 1.1], where he takes the average over $(m,t) \in \{0,\dots,[T]\} \times [0,T]$. He conjectures that the mean value theorem should hold in case of $(m,t) \in \{0,\dots,M\} \times [0,T]$. We see from the proof of our Theorem 1.1 that Duke's conjecture would imply the universality in the m-aspect.

(d) The Grossencharacter-aspect is also considered in a different context. Petridis and Sarnak [PS] obtain a subconvexity estimate of automorphic L-functions $L(s,\phi)$ for a Maass cusp form ϕ of $SL(2,Z[i])$. In order to prove it they consider the twists with Grossencharacters and take an average $\sum\int |L((1/2)+it,\phi\otimes\lambda^m)|^2 dt$, where the summation and the integration is taken over certain range of (m,t). Consequently they succeed in obtaining subconvexities in the both m and t aspects.

2. Propositions. For describing the proof of our main result, we put for $z > 0$

$$L_z(s,\lambda^m) := \prod_{N(\mathfrak{p})\le z} \left(1 - \frac{\lambda^m(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1},$$

where $\mathfrak{p}$ denotes a prime ideal. Theorem 1.1 is an immediate consequence of the following propositions:

Proposition 2.1. *For any $\varepsilon > 0$ there exists $z_0 > 0$ such that for any $z > z_0$*

$$\max_{s\in C} |\log L(s+it,\lambda^m) - \log L_z(s+it,\lambda^m)| < \varepsilon$$

holds as $T\to\infty$ for any (t,m) in a subset of $[0,T]\times\{0,\dots,[T]\}$ with positive density which is greater than $1-\varepsilon$.

Proposition 2.2. *For any $\varepsilon > 0$ there exists $z_1 > 0$ such that for any $z > z_1$*

$$\max_{s\in C} |\log L_z(s+it,\lambda^m) - \log f(s)| < \varepsilon$$

holds as $T\to\infty$ for any (t,m) in a subset of $[0,T]\times\{0,\dots,[T]\}$ with positive density which depends only on ε.

Since the intersection of the sets of (t,m) in Propositions 2.1 and 2.2 has a positive density, Theorem 1.1 follows.

3. Proof of Proposition 2.1. Put $a_m(n)$ to be the coefficient in the Dirichlet series expansion of $L(s,\lambda^m)$: $L(s,\lambda^m) = \sum_{n=1}^{\infty} a_m(n) n^{-s}$. We use the following approximate functional equation of Ramachandra type:

Lemma 3.1. *For $s = \sigma + it$ and $x, y > 0$, $xy = t^2$, under the conditions that $\sigma < \alpha < 2$, $0 < \beta < \sigma$, $0 < \gamma < 2$, we have*

$$L(s,\lambda^m) = A + B + J_1 + J_2 - \frac{W(m)}{2\pi i}\pi^{2s-1}(J_3+J_4),$$

where $|W(m)| = 1$ and

$$A = \sum_{n\le x} \frac{a_m(n)}{n^s},$$

$$B = W(m)\pi^{2s-1}\frac{\Gamma(1-s+2m)}{\Gamma(s+2m)}\sum_{n\le y}\frac{\overline{a_m(n)}}{n^{1-s}},$$

$$J_1 = \frac{1}{2\pi i}\int_{(-\gamma)} x^w \frac{\Gamma(1+\frac{w}{2})}{w}\sum_{n\le x}\frac{a_m(n)}{n^{s+w}}dw,$$

$$J_2 = \sum_{n>x}\frac{a_m(n)}{n^s}e^{-(n/x)^2},$$

$$J_3 = \frac{1}{2\pi i}\int_{(\beta)} (\pi^2 x)^w \frac{\Gamma(1-s-w+2m)}{\Gamma(s+w+2m)}\frac{\Gamma(1+\frac{w}{2})}{w} \times \sum_{n\le y}\frac{\overline{a_m(n)}}{n^{1-s-w}}dw,$$

$$J_4 = \frac{1}{2\pi i}\int_{(-\alpha)} (\pi^2 x)^w \frac{\Gamma(1-s-w+2m)}{\Gamma(s+w+2m)}\frac{\Gamma(1+\frac{w}{2})}{w} \times \sum_{n> y}\frac{\overline{a_m(n)}}{n^{1-s-w}}dw.$$

Let C_1 be a compact set in $\{s\in\mathbf{C} \mid (1/2) < \sigma < 1\}$ such that $C \subset C_1$. We will compute the integral

$$I = \sum_{m=T}^{2T}\int_T^{2T}\iint_{C_1}\left|\frac{L(s+it,\lambda^m)}{L_z(s+it,\lambda^m)} - 1\right|^2 d\sigma\, d\tau\, dt.$$

By changing the order of the integration and the sum, it follows that

$$I = \iint_{C_1}\sum_{m=T}^{2T}\int_T^{2T}\left|\frac{L(s+it,\lambda^m)}{L_z(s+it,\lambda^m)} - 1\right|^2 dt\, d\sigma\, d\tau.$$

By Lemma 3.1 we have

(3.1)

$$\sum_{m=T}^{2T}\int_T^{2T}\left|\frac{L(s+it,\lambda^m)}{L_z(s+it,\lambda^m)} - 1\right|^2 dt = \sum_{m=T}^{2T}\int_T^{2T} \left|\frac{A+B+J_1+J_2-\frac{W(m)}{2\pi i}\pi^{2s-1}(J_3+J_4)}{L_z(s+it,\lambda^m)} - 1\right|^2 dt$$

$$\ll \sum_{m=T}^{2T} \int_T^{2T} \left| \frac{A}{L_z(s+it,\lambda^m)} - 1 \right|^2 dt$$
$$+ \sum_{m=T}^{2T} \int_T^{2T} \left| \frac{B}{L_z(s+it,\lambda^m)} \right|^2 dt$$
$$+ \cdots$$
$$+ \sum_{m=T}^{2T} \int_T^{2T} \left| \frac{W(m)}{2\pi i} \pi^{2s-1} \frac{J_4}{L_z(s+it,\lambda^m)} \right|^2 dt.$$

We will compute each term in (3.1) which we put as I_A, I_B, $I_{J_1}, \ldots, I_{J_4}$. By putting $x = T$ we have for some coefficients $b_m(n)$ with $|b_m(n)| < n^\varepsilon$ such that

$$L_z(s,\lambda^m)^{-1} \sum_{n \le T} \frac{a_m(n)}{n^s} = 1 + \sum_{z<n<z^\varepsilon T} \frac{b_m(n)}{n^s}.$$

Thus

$$(3.2) \qquad I_A = \sum_{m=T}^{2T} \int_T^{2T} \left| \frac{A}{L_z(s+it,\lambda^m)} - 1 \right|^2 dt$$
$$= \sum_{m=T}^{2T} \int_T^{2T} \left| \sum_{z<n<z^\varepsilon T} \frac{b_m(n)}{n^s} \right|^2 dt.$$

By the theorem of Montgomery-Vaughn, (3.2) is estimated by

$$(3.3) \qquad T \left(T \sum_{z<n<z^\varepsilon T} \frac{1}{n^{2\sigma-\varepsilon}} + \sum_{z<n<z^\varepsilon T} \frac{1}{n^{2\sigma-\varepsilon-1}} \right)$$
$$\ll T^2 (z^{1-2\sigma+\varepsilon} + T^{1-2\sigma+\varepsilon}).$$

The contribution I_B from the term B to (3.1) is computed by using Stirling's formula as

$$(3.4) \qquad I_B \ll T^{3-2\sigma+\varepsilon}.$$

The third term I_{J_1} from J_1 is dealt with by our using the Cauchy inequality as

$$(3.5) \qquad I_{J_1} \ll T^{3-2\sigma+\varepsilon}.$$

The remaining terms $I_{J_2}, \ldots, I_{J_4}$ are similarly estimated. Taking (3.3), (3.4), (3.5) into account we have

$$\sum_{m=T}^{2T} \iint_{C_1} \int_T^{2T} \left| \frac{L(s+it,\lambda^m)}{L_z(s+it,\lambda^m)} - 1 \right|^2 dt\, d\sigma\, d\tau$$
$$\ll_{C_1} T^2 (z^{1-2\sigma_1+\varepsilon} + T^{1-2\sigma_1+\varepsilon}),$$

where $\sigma_1 = \min\{\sigma \in C_1\}$. Since $\sigma_1 > (1/2)$, by taking z_0 as $z_0^{1-2\sigma_1+\varepsilon} = \varepsilon^3$, we have

$$(3.6) \qquad \frac{1}{T^2} \sum_{m=T}^{2T} \int_T^{2T}$$
$$\left(\iint_{C_1} \left| \frac{L(s+it,\lambda^m)}{L_z(s+it,\lambda^m)} - 1 \right|^2 d\sigma\, d\tau \right) dt < \varepsilon^3$$

for $z > z_0$, $T > T_0(z)$. It follows from (3.6) that there exists a subset A_T of $[0,T] \times \{0,\ldots,[T]\}$ with positive density greater than $1-\varepsilon$ such that

$$\iint_{C_1} \left| \frac{L(s+it,\lambda^m)}{L_z(s+it,\lambda^m)} - 1 \right|^2 d\sigma\, d\tau < \varepsilon^2$$

for any $(t,m) \in A_T$. We then have

$$\max_{s \in C} \left| \frac{L(s+it,\lambda^m)}{L_z(s+it,\lambda^m)} - 1 \right| \ll_{C,C_1} \varepsilon.$$

This means that

$$\max_{s \in C} |\log L(s+it,\lambda^m) - \log L_z(s+it,\lambda^m)| \ll_{C,C_1} \varepsilon$$

for $(t,m) \in A_T$.

Remark 3.2. Duke's conjecture [D] would make it possible to deal with the variables m and t separately.

4. Proof of Proposition 2.2.

Lemma 4.1 (Gonek [G])**.** *Let C be a simply connected compact set of the strip $(1/2) < \sigma < 1$. Let $h(s)$ be a continuous function on C which is regular on $\mathrm{Int}(C)$. For any $y > 0$ there exist $\nu_0 = \nu_0(C,h,y)$ and $\theta_p^{(0)} \in [0,1]$ such that*

$$\max_{s \in C} \left| h(s) - \sum_{\substack{y<p\le\nu \\ p\equiv 1 \pmod 4}} \frac{e(\theta_p^{(0)})}{p^s} \right| \ll_C y^{-\frac{1}{2}}$$

for any $\nu > \nu_0$, where p denotes the prime numbers.

Lemma 4.2 ([KV] Theorems 8.1, 8.2)**.** *Let $a_n \in \mathbf{R}$ $(1 \le n \le N)$ be linearly independent over $\mathbf{Q}$. Then we have*

(i) *If we put*

$$I_A(T) := \{t \in [0,T] \mid (\{a_1 t\},\ldots,\{a_N t\}) \in A\}$$

for any closed Jordan measurable set $A \subset [0,1]^N$ and for $T > 0$, where $\{x\} = x - [x]$, it holds that $\lim_{T\to\infty}(\mu(I_A(T))/T) = \mu_N(A)$ with μ_N the Lebesgue measure on $\mathbf{R}^N$.

(ii) *Let Ω be a set of continuous functions on A. If Ω is uniformly bounded and is equicontinuous, it holds uniformly on $f \in \Omega$ that*

$$\lim_{T\to\infty}\frac{1}{T}\int_{I_A(T)} f(\{a_1t\},\ldots,\{a_Nt\})dt = \int\cdots\int_A f(x_1,\ldots,x_N)dx_1\cdots dx_N.$$

Lemma 4.3. *Let $p\equiv 1 \pmod 4$ and $(p)=\mathfrak{p}\overline{\mathfrak{p}}$ with $\mathfrak{p}$ a prime ideal in K. We put θ_p as $\lambda(\mathfrak{p})=e^{i\theta_p}$. Then $\{\theta_p\}_{p\equiv 1 \pmod 4}$ is linearly independent over $\mathbf{Q}$.*

Proof. Putting $\mathfrak{p}=(a+bi)$ $(a,b\in\mathbf{Z})$, we have $|\alpha|=\sqrt{p}$ and so $\lambda(\mathfrak{p})=((a+bi)/\sqrt{p})^4$. Thus $\cos\theta_p$, $\sin\theta_p\in\mathbf{Z}[1/\sqrt{p}]$. Assume an algebraic dependence as $M\theta_p=m_1\theta_{p_1}+\cdots+m_r\theta_{p_r}$ with $M,m_1,\ldots,m_r\in\mathbf{Z}$. Then in the equation $\cos(M\theta_p)=\cos(m_1\theta_{p_1}+\cdots+m_r\theta_{p_r})$, the left hand side belongs to $\mathbf{Z}[1/\sqrt{p}]$, whereas the right hand side is in $\mathbf{Z}[1/\sqrt{p_1},\ldots,1/\sqrt{p_r}]$. Hence it holds if and only if $\cos(M\theta_p)\in\mathbf{Z}$. Therefore we have $M=0$. □

Proof of Proposition 2.2. We have

$$\log L_z(s,\lambda^m)=\sum_{\substack{p\le z\\ p\equiv 1 \pmod 4}}\sum_{k=1}^{\infty}\frac{2\cos(km\theta_p)}{kp^s} + \sum_{\substack{p\le z\\ p\equiv 3 \pmod 4}}\sum_{k=1}^{\infty}\frac{1}{kp^{2ks}}+\sum_{k=1}^{\infty}\frac{1}{k2^{ks}}.$$

We split the sums over $p\le z$ into the ones over $p\le y$ and $y<p\le z$ with $0<y<z$. We also divide the sum over $1\le k<\infty$ into $k=1$, $2\le k<N$, and $k\ge N$ with $N=[\sigma\log_2 y]$. For partial sums we have the estimates $\sum_{y<p\le z}\sum_{2\le k<N}(2\cos(km\theta_p)/kp^s)\ll y^{1-2\sigma}$ and $\sum_{p\le y}\sum_{k\ge N}(2\cos(km\theta_p)/kp^s)\ll y2^{-N\sigma}\ll y^{1-2\sigma}$. Hence

(4.1)
$$\log L_z(s+it,\lambda^m)=\sum_{\substack{y<p\le z\\ p\equiv 1 \pmod 4}}\frac{2\cos(m\theta_p)}{p^s} + l(s+it,y,m)+O(y^{1-2\sigma}),$$

where

(4.2)
$$l(s,y,m)=\sum_{\substack{p\le y\\ p\equiv 1 \pmod 4}}\sum_{k\le N}\frac{2\cos(km\theta_p)}{kp^{ks}} + \sum_{\substack{p^2\le y\\ p\equiv 3 \pmod 4}}\sum_{k\le N}\frac{1}{kp^{2ks}}+\sum_{k\le N}\frac{1}{k2^{ks}}.$$

We fix sufficiently large y which satisfies $y^{1-2\sigma}<\varepsilon$ and $y^{-(1/2)}<\varepsilon$. Apply Lemma 4.1 for $h(s)=(1/2)(g(s)-l(s,y,0))$ and fix $\nu>\nu_0$. Then for any $z>\nu$,

$$|\log L_z(s+it,\lambda^m)-g(s)|$$

$$(4.3)\quad \le\left|\sum_{\substack{y<p\le\nu\\ p\equiv 1 \pmod 4}}\frac{2\cos(m\theta_p)}{p^{s+it}}-\sum_{y<p\le\nu}\frac{2e(\theta_p^{(0)})}{p^s}\right|$$

$$(4.4)\quad +|l(s+it,y,m)-l(s,y,0)|$$

$$(4.5)\quad +\left|\sum_{\substack{\nu<p\le z\\ p\equiv 1 \pmod 4}}\frac{2\cos(m\theta_p)}{p^{s+it}}\right|+\varepsilon.$$

We first deal with (4.3). It is less than

$$(4.6)\quad \sum_{\substack{y<p\le\nu\\ p\equiv 1 \pmod 4}}\frac{2}{p^\sigma}\left|\frac{\cos(m\theta_p)}{p^{it}}-e(\theta_p^{(0)})\right| = \sum_{\substack{y<p\le\nu\\ p\equiv 1 \pmod 4}}\frac{2}{p^\sigma}\left|\cos(m\theta_p)e^{-it\log p}-e(\theta_p^{(0)})\right|.$$

Hence if we take a sufficiently small $\delta>0$ and put

$$V_T^{(1)}=\{0\le m\le T \mid \|m\theta_p\|<\delta \quad (y<p\le\nu,\ p\equiv 1 \pmod 4)\},$$

$$U_T^{(1)}=\left\{t\in[0,T] \mid \left\|t\frac{\log p}{2\pi}-\theta_p^{(0)}\right\|<\delta \quad (y<p\le\nu,\ p\equiv 1 \pmod 4)\right\},$$

then for any $(m,t)\in V_T^{(1)}\times U_T^{(1)}$, it holds that (4.6) $<\varepsilon$. By Lemmas 4.2, 4.3, and the linear independence over $\mathbf{Q}$ of $\{\log p\}$, we have

$$(4.7)\quad \lim_{T\to\infty}\frac{\mu'(V_T^{(1)}\times U_T^{(1)})}{T^2}=\sharp(V^{(1)})\times\mu(U^{(1)})$$

for some $V^{(1)},U^{(1)}\subset\mathbf{R}^{\pi(\nu)-\pi(y)}$ with $\pi(x)$ the number of primes not greater than x.

Next we consider (4.4). It is less than

$$(4.8)\quad \sum_{\substack{p\le y\\ p\equiv 1 \pmod 4}}\sum_{1\le k\le N}\frac{1}{kp^{k\sigma}}\left|\frac{2\cos(km\theta_p)}{p^{ikt}}-2\right| + \sum_{\substack{p^2\le y\\ p\equiv 3 \pmod 4}}\sum_{1\le k\le N}\frac{1}{kp^{2k\sigma}}\left|\frac{1}{p^{2ikt}}-1\right| + \sum_{1\le k\le N}\frac{1}{2^{k\sigma}}\left|\frac{1}{2^{ikt}}-1\right|.$$

Again we take a sufficiently small $\delta' > 0$ and put

$$V_T^{(2)} = \{0 \le m \le T \mid \|m\theta_p\| < \delta' \quad (p \le y,\ p \equiv 1 \pmod 4)\},$$

$$U_T^{(2)} = \Big\{t \in [0,T] \mid \Big\|t\frac{\log p}{2\pi}\Big\| < \delta'(p \le y)\Big\}.$$

Then for any $(m,t) \in V_T^{(2)} \times U_T^{(2)}$, it holds that $(4.8) < \varepsilon$.

We put

$$V_T = \{0 \le m \le T \mid \|m\theta_p\| < \delta\ (y < p \le \nu,\ p \equiv 1 \pmod 4),\ \|m\theta_p\| < \delta'\ (p \le y,\ p \equiv 1 \pmod 4)\}$$

and

$$U_T = \{t \in [0,T] \mid \Big\|t\frac{\log p}{2\pi} - \theta_p^{(0)}\Big\| < \delta\ (y < p \le \nu,\ p \equiv 1 \pmod 4),\ \Big\|t\frac{\log p}{2\pi}\Big\| < \delta'\ (p \le y)\}.$$

Then (4.3) and (4.4) are bounded by ε for any $(m,t) \in V_T \times U_T$, and we have

$$\lim_{T\to\infty} \frac{\sharp V_T}{T} = \mathrm{vol}(V) = (2\delta)^{\frac{\pi(\nu)-\pi(y)}{2}}(2\delta')^{\frac{\pi(y)}{2}}$$

$$\lim_{T\to\infty} \frac{\mu(U_T)}{T} = \mathrm{vol}(U) = (2\delta)^{\pi(\nu)-\pi(y)}(2\delta')^{\pi(y)},$$

where U and V are subsets of $[0,1]^{\pi(\nu)}$. Here we have proved that (4.3) and (4.4) are less than ε for any (m,t) in a set with positive density.

Lastly we can check that (4.5) is less than ε for almost all $(m,t) \in U_T \times V_T$. This completes the proof of the theorem. □

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