On critical Riemannian metrics for a curvature functional on 3-manifolds

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Abstract: The normalized L^2-norm of the traceless part of the Ricci curvature defines a Riemannian functional on the space of metrics. In this paper, we will consider this functional on 3-manifolds.

Key words: Critical metrics; Riemannian functionals.

1. Introduction. Let M be a closed oriented smooth 3-manifold, $\mathcal{M}(M)$ the space of smooth Riemannian metrics on M, and $\mathrm{Diff}(M)$ the diffeomorphism group. We consider a functional $F : \mathcal{M}(M) \to \boldsymbol{R}$ defined by

(1.1)
$$F(g) = \left(\int_M dv_g\right)^{1/3} \int_M \left|\mathrm{Ric}_g - \frac{R_g}{3} g\right|^2 dv_g = \left(\int_M dv_g\right)^{1/3} \int_M |Z_g|^2 dv_g,$$

where dv_g, Ric_g, R_g, and $Z_g := \mathrm{Ric}_g - (1/3)R_g g$ denote the volume element, the Ricci curvature, the scalar curvature, and the traceless part of the Ricci curvature of (M, g), respectively. The functional F is Riemannian, that is, for all $g \in \mathcal{M}(M)$, $\varphi \in \mathrm{Diff}(M)$, and any positive constant c, $F(\varphi^* g) = F(cg) = F(g)$.

Obviously, if g is an Einstein metric (a constant curvature metric in dimension 3), then g is a critical point for F. It is well known that a critical point of the total scalar curvature functional $I : \mathcal{M}(M) \to \boldsymbol{R}$ defined by

$$I(g) = \left(\int_M dv_g\right)^{-1/3} \int_M R_g dv_g \tag{1.2}$$

is exactly an Einstein metric. However, it is difficult to obtain an existence theory for critical points for the total scalar curvature functional I. Since the Riemann curvature is needed to control the convergence or degeneration of metrics, the scalar curvature alone being too weak, it is perhaps natural to consider other Riemannian functionals ([1]). In dimension 3, it seems that one can control the Riemann curvature of a critical metric for the functional F. On the other hand, we do not know complete geometric properties of critical metrics for the functional F.

In this paper, we will construct a critical metric for F on the 3-sphere S^3 which is not of Einstein, and prove that a critical point must be an Einstein metric under some conditions.

2. Preliminaries.

Lemma 2.1. *Let $\{g(t)\} \subset \mathcal{M}(M)$ be a one-parameter family of metrics with $g(0) = g$ and $(d/dt)g(t)|_{t=0} = h$. Then*

$$\frac{d}{dt}\int_M |Z_{g(t)}|^2 dv_{g(t)}\Big|_{t=0} = \int_M (h, L_g)_g \, dv_g, \tag{2.1}$$

$$L_g := \bar{\Delta}_g Z_g + \frac{1}{3}\nabla^2 R_g + \frac{1}{6}(\Delta_g R_g) g + 4 Z_g \cdot Z_g - \frac{3}{2}|Z_g|^2 g + \frac{1}{3} R_g Z_g, \tag{2.2}$$

where $\bar{\Delta}_g = \delta_g \nabla = -\nabla^k \nabla_k$ and $\Delta_g = \delta_g d = -\nabla^k \nabla_k$ is the rough Laplacian and the Laplacian, and $(Z_g \cdot Z_g)_{ij} = Z_{ik} Z_{jl} g^{kl}$.

Proof. Direct calculations, see also [6]. □

Corollary 2.2. *A Riemannian metric $g \in \mathcal{M}(M)$ is critical for F if and only if g satisfies the following Euler-Lagrange equations:*

(2.3)
$$L_g^1 := \bar{\Delta}_g Z_g + \frac{1}{3}\left(\nabla^2 R_g + \frac{1}{3}(\Delta_g R_g) g\right) + 4\left(Z_g \cdot Z_g - \frac{1}{3}|Z_g|^2 g\right) + \frac{1}{3} R_g Z_g = 0,$$

$$L_g^2 := -\Delta_g R_g + 3|Z_g|^2 - c = 0, \tag{2.4}$$

where $c = 3F(g)/\mathrm{Vol}(M, g)$ is the non-negative constant.

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Proof. From the Lagrange multiplier argument, we have $L_g = \lambda g$ for some constant λ. Taking trace of this equation, we have $L_g^2 = 0$, and so $L_g^1 = 0$. Integrating the equation $L_g^2 = 0$ over M, we get $c = 3F(g)/\operatorname{Vol}(M, g)$. □

Remark 2.3. The gradient vector $(\nabla F)_g$ of F at $g \in \mathcal{M}(M)$ is in the tangent space $T_g\mathcal{M}(M) \cong \Gamma(S^2T^*M)$, the space of symmetric tensors of type (0,2). The space $T_g\mathcal{M}(M)$ has the trivial pointwise orthogonal splitting with respect to g:

$$T_g\mathcal{M}(M) \cong \Gamma_g^T(S^2T^*M) \oplus C^\infty(M) \cdot g, \tag{2.5}$$

where $\Gamma_g^T(S^2T^*M)$ is the space of traceless symmetric tensors of type (0,2), and $C^\infty(M)$ is the space of smooth functions on M. Up to constant, L_g^1 and L_g^2 are orthogonal projection of $(\nabla F)_g$ on these spaces.

3. Critical metrics on S^3.

Theorem 3.1. *There is a critical Riemannian metric g on S^3 for F which is of Berger type but not of Einstein.*

We consider the Lie group $SU(2) \cong S^3$, and choose the following basis of the Lie algebra of $SU(2)$,

$$\left\{ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}. \tag{3.1}$$

Let $\{e_1, e_2, e_3\}$ denote the corresponding basis of left-invariant vector fields. Define

$$g(t) = (g(t)(e_i, e_j)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^2 \end{pmatrix} \tag{3.2}$$

to be an inner product on the Lie algebra of $SU(2)$ written in the basis $\{e_1, e_2, e_3\}$. We then have

$$\begin{aligned} |Z_{g(t)}|^2 &= |\operatorname{Ric}_{g(t)}|^2 - \frac{1}{3}|R_{g(t)}|^2 \\ &= (32 - 32t^2 + 12t^4) \qquad (3.3) \\ &\quad - \frac{1}{3}(64 - 32t^2 + 4t^4) \\ &= \frac{32}{3}(1 - 2t^2 + t^4), \qquad (3.4) \end{aligned}$$

and

$$\operatorname{Vol}(SU(2), g(t)) = \frac{1}{2\sqrt{2}} V_0 t, \tag{3.5}$$

where V_0 is the volume of $SU(2)$ induced by its Killing form. Therefore

(3.6)
$$\begin{aligned} F(g(t)) &= \operatorname{Vol}(SU(2), g(t))^{4/3} |Z_{g(t)}|^2 \\ &= \frac{32}{3} \left(\frac{V_0}{2\sqrt{2}} \right)^{4/3} t^{4/3}(1 - 2t^2 + t^4). \end{aligned}$$

Using the symmetric criticality principle ([5], see also [3, 4]), it follows that a Riemannian metric is critical for all variations if and only if it is critical for the above one-parameter family. Simple calculation shows that $t^2 = 1$ and $t^2 = 1/4$ are critical points for F. The first case corresponds to the standard round sphere, and the second case is a metric of Berger type.

Remark 3.2. In [6], Tanno proved that if a positive constant scalar curvature metric g with $|Z_g|^2 \le (1/26)R_g^2$ is critical for F, then g must be the Einstein metric.

4. Critical metrics with a flat conformal structure.

Theorem 4.1. *Let M be a closed 3-manifold, and $g \in \mathcal{M}(M)$ a critical Riemannian metric for F with a flat conformal structure. If $\int_M R_g dv_g \ge 0$, then g is an Einstein metric.*

Lemma 4.2. *Let (M, g) be a conformally flat 3-manifold. Then*

(4.1)
$$\begin{aligned} &\bar{\Delta}_g Z_g + \frac{1}{4}\left(\nabla^2 R_g + \frac{1}{3}(\Delta_g R_g) g \right) \\ &\qquad + 3\left(Z_g \cdot Z_g - \frac{1}{3}|Z_g|^2 g \right) + \frac{1}{2} R_g Z_g = 0. \end{aligned}$$

Proof. Since (M, g) is conformally flat, the Cotten tensor vanishes, i.e.,

$$\begin{aligned} &\nabla_k Z_{ij} - \nabla_j Z_{ik} \\ &\qquad + \frac{1}{12}(\nabla_k R g_{ij} - \nabla_j R g_{ik}) = 0. \end{aligned} \tag{4.2}$$

Differentiating and contracting this equation and from the second Bianchi identity, we have

(4.3)
$$\begin{aligned} 0 &= -\bar{\Delta} Z_{ij} - \nabla^k \nabla_j Z_{ik} \\ &\quad + \frac{1}{12}(-\Delta R g_{ij} - \nabla_i \nabla_j R) \\ &= -\bar{\Delta} Z_{ij} - \nabla_j \nabla_k Z_i^k - R_{lkj}^k Z_i^l + R_{ikj}^l Z_l^k \\ &\quad + \frac{1}{12}(-\Delta R g_{ij} - \nabla_i \nabla_j R) \\ &= -\bar{\Delta} Z_{ij} - \frac{1}{6}\nabla_i \nabla_j R - R_{lj} Z_i^l + R_{ikj}^l Z_l^k \\ &\quad + \frac{1}{12}(-\Delta R g_{ij} - \nabla_i \nabla_j R). \end{aligned}$$

Using

$$R^l_{ikj} = \delta^l_k Z_{ij} - \delta^l_j Z_{ik} + Z^l_k g_{ij} - Z^l_j g_{ik} + \frac{R}{6}(\delta^l_k g_{ij} - \delta^l_j g_{ik}), \tag{4.4}$$

we get the desired result. □

Proof of the theorem. Because that g is critical and conformally flat, we have

$$\bar{\Delta}_g Z_g + \frac{1}{3}\left(\nabla^2 R_g + \frac{1}{3}(\Delta_g R_g) g\right) + 4\left(Z_g \cdot Z_g - \frac{1}{3}|Z_g|^2 g\right) + \frac{1}{3} R_g Z_g = 0, \tag{4.5}$$

$$\bar{\Delta}_g Z_g + \frac{1}{4}\left(\nabla^2 R_g + \frac{1}{3}(\Delta_g R_g) g\right) + 3\left(Z_g \cdot Z_g - \frac{1}{3}|Z_g|^2 g\right) + \frac{1}{2} R_g Z_g = 0. \tag{4.6}$$

From these two equations

$$\bar{\Delta}_g Z_g + R_g Z_g = 0. \tag{4.7}$$

Taking inner product of this equation with Z_g and integral by parts, we have

$$\int_M (|\nabla Z_g|^2 + R_g |Z_g|^2) dv_g = 0. \tag{4.8}$$

Multipling the equation $L^2_g = 0$ by R_g, and integrating the result over M, we get

$$\int_M R_g |Z_g|^2 dv_g = \frac{1}{3}\int_M (|\nabla R_g|^2 + cR_g) dv_g. \tag{4.9}$$

Therefore

$$\int_M \left(|\nabla Z_g|^2 + \frac{1}{3}|\nabla R_g|^2\right) dv_g = -\frac{c}{3}\int_M R_g dv_g \leq 0, \tag{4.10}$$

and we know that R_g is non-negative constant.

If $R_g > 0$, then $c = 3F(g)/\operatorname{Vol}(M, g) = 0$, and g is an Einstein metric of the spherical space form.

If $R_g = 0$, then $Z_g \cdot Z_g = 1/3|Z_g|^2 g$. We choose a local frame of orthonormal vector fields adopted to g such that $Z_{ij} = \lambda_i \delta_{ij}$. The values of λ_i are the eigenvalues of the traceless part of the Ricci curvature of g. Simple linear algebraic argument shows that $\lambda_1 + \lambda_2 + \lambda_3 = 0$, and $\lambda_1^2 = \lambda_2^2 = \lambda_3^2$. Therefore $\lambda_1 = \lambda_2 = \lambda_3 = 0$, that is $Z_g = 0$, and g is an Einstein metric of the Euclidean space form. □

References

[1] Anderson, M. T.: Extrema of curvature functionals on the space of metrics on 3-manifolds. Calc. Var. PDE, **5**, 199–269 (1997).

[2] Besse, A. L.: Einstein Manifolds. Erg. Math. Grenzgeb., Bd. 10, Springer, Berlin-Heidelberg-New York, pp. 1–510 (1987).

[3] Gursky, M. J., and Viaclovsky, J. A.: A new variational characterization of three-dimensional space form. Invent. Math., **145**, 251–278 (2001).

[4] Lamontagne, F.: A critical metric for the L^2-norm of the curvature tensor on S^3. Proc. Amer. Math. Soc., **126**, 589–593 (1998).

[5] Palais, R. S.: The principle of symmetric criticality. Comm. Math. Phys., **69**, 19–30 (1979).

[6] Tanno, S.: Deformations of Riemannian metrics on 3-dimensional manifolds. Tôhoku Math. J., **27**, 437–444 (1975).