Some invariant and equivariant cohomology classes of the space of Kähler metrics

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Abstract: Invariant and equivariant cohomology classes on the space of Kähler forms are defined. Relations to the obstructions to the existence of Kähler-Einstein metrics and Kähler metrics of harmonic Chern forms are discussed.

Key words: Kähler forms; invariant cohomology; equivariant cohomology; Hermitian multiplier structure; Kähler-Einstein metric.

1. Multiplier Hermitian structure case. In this paper we consider the action on subspaces of the space of all Kähler forms by subgroups of the automorphism group $\mathrm{Aut}(M)$ of an m-dimensional compact Kähler manifold M, and study some invariant and equivariant de Rham cohomology classes and also Alexander-Spanier cohomology classes of the space of Kähler forms. Here we mean by invariant cohomology groups those of the subcomplexes of invariant cochains.

If we fix a Kähler metric g_0 with Kähler form ω_0 on M, the space Ω of all Kähler forms in the cohomology class $[\omega_0]$ is described as

$$\Omega = \{\omega = \omega_0 + i\partial\overline{\partial}\varphi > 0 \mid \varphi \in C^\infty(M)\},$$

where $C^\infty(M)$ denotes the set of all real valued smooth functions on M.

First of all we assume $\Omega = c_1(M) > 0$, and consider the multiplier Hermitian structure introduced by Mabuchi [13]. Put

$$\Omega_Y = \{\omega \in \Omega \mid L_{Y_{\mathbf{R}}}\omega = 0\},$$

where $Y_{\mathbf{R}} = Y + \overline{Y}$ is the real part of a holomorphic vector field Y on M. Note that $\Omega_Y = \Omega$ in the case where $Y = 0$. We assume that $\Omega_Y \neq \emptyset$ and that Y is a Hamiltonian, i.e. for all

$$\omega = i\sum_{\alpha,\beta=1}^{m} g_{\alpha\overline{\beta}}\, dz^\alpha \wedge d\overline{z}^\beta \in \Omega_Y$$

there associates a function $u_\omega \in C^\infty(M)$ such that

$$Y = i\,\mathrm{grad}_\omega\, u := i\sum_{\alpha,\beta=1}^{m} g^{\alpha\overline{\beta}}\frac{\partial u_\omega}{\partial \overline{z}^\beta}\frac{\partial}{\partial z^\alpha},$$

where $z^1, \dots, z^m$ denote local holomorphic coordinates of M. Under the normalization $\int_M u_\omega \omega^m = 0$ the image $I_Y \subset \mathbf{R}$ of u_ω is independent of $\omega \in \Omega_Y$. Choose an arbitrary smooth function σ on I_Y and consider a smooth function $\psi_\omega = \sigma(u_\omega)$ on M. A multiplier Hermitian structure is by definition an assignment of the Hermitian structure $\widetilde{\omega} = e^{-\psi_\omega/m}\omega$. This structure was introduced to study those Kähler metrics, which we shall call σ-Kähler-Einstein metrics, satisfying the equation $\mathrm{Ric}(\widetilde{\omega}) = \omega$ where

$$\mathrm{Ric}(\widetilde{\omega}) = -i\partial\overline{\partial}\log\widetilde{\omega}^m = -i\partial\overline{\partial}\log\left(e^{-\psi_\omega}\det(g_{\alpha\overline{\beta}})\right).$$

The σ-Kähler-Einstein metrics include Kähler-Ricci solitons [14] and Kähler-Einstein metrics in Mabuchi's sense [12] as special cases. Define F_ω, $\widetilde{F}_\omega \in C^\infty(M)$, which are defined up to a constant, by

$$\mathrm{Ric}(\omega) - \omega = i\partial\overline{\partial}F_\omega, \quad \widetilde{F}_\omega = F_\omega + \psi_\omega.$$

Then ω is a σ-Kähler-Einstein metric if and only if $\widetilde{F}_\omega$ is constant.

Let Z_Y be the subgroup of $\mathrm{Aut}(M)$ consisting of all elements g such that $\mathrm{Ad}(g)Y = Y$, and let $\mathfrak{z}_Y$ denote the Lie algebra of Z_Y. Then Z_Y acts on Ω_Y. We define a linear functional $f_Y : \mathfrak{z}_Y \to \mathbf{C}$ by

$$f_Y(X) = \int_M X(\widetilde{F}_\omega)\widetilde{\omega}^m.$$

By a proof similar to [7] one can show that f_Y is independent of the choice of $\omega \in \Omega_Y$ and define a character of the Lie algebra $\mathfrak{z}_Y$.

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We define a smooth 1-form α_Y on Ω_Y by

$$\alpha_Y = \widetilde{\Delta}\widetilde{F}_\omega \widetilde{\omega}^m,$$

where $\widetilde{\Delta} = e^{\psi_\omega}\overline{\partial}^*(e^{-\psi_\omega}\overline{\partial})$.

Theorem 1.1. (1) *The* 1*-form* α_Y *defines a* 1*-dimensional* Z_Y*-invariant cohomology class of* Ω_Y*. Moreover if the character* f_Y *is nontrivial then the invariant cohomology class* $[\alpha_Y]$ *is nontrivial.*

(2) *If the character* f_Y *is trivial then* α_Y *is a basic form in the Weil model of* Z_Y*-equivariant cohomology* (*cf.* [4, 11])*, but it defines a trivial equivariant de Rham class in the Weil model.*

Consider a smooth path ω_t, $a \le t \le b$, in Ω_Y with

$$\omega_t = \omega_0 + i\partial\overline{\partial}\phi_t, \quad \omega_0 \in \Omega_Y, \quad \phi_t \in C^\infty(M),$$

and put (cf. [13])

$$M^\sigma(\omega_a, \omega_b) = \int_a^b \int_M \dot{\phi}_t \widetilde{\Delta}\widetilde{F}_{\omega_t}\widetilde{\omega}_t^m.$$

M^σ is independent of the choice of the path $\{\phi_t\}$, and is called the *K-energy map*. It is proved in [13] that if there exists a σ-Kähler-Einstein form ω_{KE} then the subspace of Ω_Y consisting of all σ-Kähler-Einstein forms is the orbit of ω_{KE} by the action of the identity component Z_Y^0 of Z_Y. This in particular implies that $Z_Y \cdot \omega_{KE} = Z_Y^0 \cdot \omega_{KE}$. Note that M^σ is Z_Y-invariant, satisfies cocycle conditions and therefore determines an element of Z_Y-invariant Alexander-Spanier cohomology. See [15] for Alexander-Spanier cohomologies.

Theorem 1.2. *If* $f_Y = 0$ *then* M^σ *defines a trivial* 1*-dimensional class in the* Z_Y*-invariant Alexander-Spanier cohomology of* Ω_Y*.*

Proof of Theorem 1.1. (1) The first statement follows from direct computations (see Bourguignon [3] for the case $Y = 0$ and $\sigma = 0$). To prove the second statement suppose that $f_Y(X) \neq 0$ for $X \in \mathfrak{z}_Y$ and that $\alpha_Y = d\beta$ for some Z_Y-invariant function β. The Z_Y-invariance of β implies $X^\sharp\beta = 0$ where $X^\sharp$ denotes the vector field on Ω_Y induced by $X \in \mathfrak{z}_Y$. Note that, when we regard the tangent space of Ω_Y as the space of $Y_{\mathbf{R}}$-invariant functions modulo constants, $X^\sharp$ is equal to the divergence of X. From this we have

$$X^\sharp\beta = i(X^\sharp)d\beta = i(X^\sharp)\alpha_Y = f_Y(X) \neq 0,$$

which is a contradiction.

(2) The two conditions for the definition of basic forms correspond to the Z_Y-invariance of α_Y and the assumption $f_Y(X) = 0$. Choose a fixed $\omega_0 \in \Omega_Y$ and put $\mu(\omega) = M^\sigma(\omega_0, \omega)$. We wish to see that μ is Z_Y-invariant. But this is equivalent to

$$M^\sigma(\omega, a^*\omega) = 0$$

for any $a \in Z_Y$ and $\omega \in \Omega_Y$. The assumption $f_Y(X) = 0$ tells us that μ is Z_Y^0-invariant. But Z_Y has only finitely many components by Theorem 4.8 and Lemma 2.4 in [6], and thus for any $a \in Z_Y$ we have $a^n \in Z_Y^0$ for some n. It follows from this, the cocycle conditions and Z_Y-invariance of M^σ that

$$\begin{aligned} 0 &= M^\sigma(\omega, (a^n)^*\omega) \\ &= M^\sigma(\omega, a^*\omega) + \cdots + M^\sigma((a^{n-1})^*\omega, (a^n)^*\omega) \\ &= (n-1)M^\sigma(\omega, a^*\omega). \end{aligned}$$

Hence μ is Z_Y-invariant. Since $d\mu = \alpha_Y$ we are done. □

Proof of Theorem 1.2. As the previous proof shows if $f_Y = 0$ then μ is Z_Y-invariant. Moreover from the cocycle conditions of M^σ one sees that M^σ is a coboundary of μ. □

2. Higher Chern class case. Following Bando [1] (see also [2]) which extends earlier works by the author([7, 8]) and Calabi([5]), we define a closed 1-form α_k on Ω as follows: Let $c_k(\omega)$ denote the k-th Chern form with respect to ω and put

$$\lambda_k = \frac{\langle c_k(M) \cup [\omega]^{m-k}, [M]\rangle}{\langle [\omega]^m, [M]\rangle}.$$

Define a 1-form α_k on Ω, the tangent space of which being identified with the space of smooth functions modulo constants, by

$$\alpha_k = c_k(\omega) \wedge \omega^{m-k} - \lambda_k\omega^m.$$

It is well-known that α_k is closed and invariant under the subgroup $\mathrm{Aut}_\Omega(M)$ of $\mathrm{Aut}(M)$ consisting of all automorphisms preserving Ω (cf. [9]).

Next we define a functional $f_k : \mathfrak{a} \to \mathbf{C}$ of the Lie algebra $\mathfrak{a}$ of all holomorphic vector fields on M into $\mathbf{C}$ by

$$f_k(X) = \int_M L_X F_k \wedge \omega^{m-k+1}.$$

Then f_k is independent of the choice of $\omega \in [\omega_0]$, and therefore $\mathrm{Aut}(M)$-invariant. In particular f_k defines a Lie algebra character.

Theorem 2.1. (1) *The* 1*-form* α_k *defines a* 1*-dimensional* $\mathrm{Aut}_\Omega(M)$*-invariant cohomology class of* Ω*. Moreover if the character* f_k *is nontrivial then the invariant cohomology class* $[\alpha_k]$ *is nontrivial.*

(2) *If the character f_k is trivial then α_k is a basic form in the Weil model of* $\mathrm{Aut}_\Omega(M)$*-equivariant cohomology of* Ω*, but it defines a trivial equivariant de Rham class in the Weil model.*

Proof. The proof of Theorem 2.1 is quite analogous to that of Theorem 1.1 if we define the k-th K-energy N_k by

$$N_k(\omega_a, \omega_b) = \int_a^b \int_M \dot{\phi}_t(c_k(\omega) \wedge \omega^{m-k} - \lambda_k \omega^m).$$

□

Theorem 2.2. *If $f_k = 0$ then N_k defines a trivial* 1*-dimensional class in the* $\mathrm{Aut}_\Omega(M)$*-invariant Alexander-Spanier cohomology of* Ω*.*

Proof. Quite analogous to the proof of Theorem 1.2. □

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