A proof of an order preserving inequality

By Takayuki FURUTA

Department of Applied Mathematics, Science University of Tokyo, 1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601

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Abstract: Simplified proof of an order preserving operator inequality is given.

Key word: Order preserving inequality.

A capital letter means a bounded linear operator on a Hilbert space. Löwner-Heinz inequality asserts:

$(*)$ $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$.

We obtain the following result in [1].

Theorem A. *If $A \geq B > 0$, then for each $t \in [0,1]$ and $p \geq 1$*

$$A^{1+r-t} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} \tag{1}$$

holds for $r \geq t$ and $s \geq 1$.

M. Uchiyama [3] shows the following interesting extension of Theorem A.

Theorem B. *If $A \geq B \geq C > 0$, then for each $t \in [0,1]$ and $p \geq 1$*

$$A^{1+r-t} \geq \{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} \tag{2}$$

holds for $r \geq t$ and $s \geq 1$.

Here we show a simplified proof of Theorem B by using Theorem A itself. We need the following result which is Lemma 1 in [1].

Lemma. *Let $X > 0$ and Y be invertible. For any real number λ*

$$(YXY^*)^\lambda = YX^{\frac{1}{2}}(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}})^{\lambda-1}X^{\frac{1}{2}}Y^*.$$

Proof of Theorem B. Put $Y = A^{\frac{t}{2}}B^{\frac{-t}{2}}$. As $A^t \geq B^t$ by $(*)$ since $t \in [0,1]$, we have by the hypotheses

$$Y^*Y = B^{\frac{-t}{2}}A^tB^{\frac{-t}{2}} \geq I \quad \text{and} \quad \lambda = \frac{1}{(p-t)s+t} \in [0,1]. \tag{3}$$

Put $D = B^{\frac{t}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sB^{\frac{t}{2}}$. As $B \geq C > 0$, we have by Theorem A for $r = t$

$$B \geq D^\lambda. \tag{4}$$

Then we have

$$\begin{aligned} B_1 &= \{A^{\frac{t}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}} \\ &= (A^{\frac{t}{2}}B^{\frac{-t}{2}}DB^{\frac{-t}{2}}A^{\frac{t}{2}})^\lambda \\ &= YD^{\frac{1}{2}}(D^{\frac{1}{2}}Y^*YD^{\frac{1}{2}})^{\lambda-1}D^{\frac{1}{2}}Y^* \text{ by Lemma} \\ &\leq YD^{\frac{1}{2}}D^{\lambda-1}D^{\frac{1}{2}}Y^* \\ &= YD^\lambda Y^* \\ &\leq A^{\frac{t}{2}}B^{\frac{-t}{2}}BB^{\frac{-t}{2}}A^{\frac{t}{2}} \text{ by (4)} \\ &= A^{\frac{t}{2}}B^{1-t}A^{\frac{t}{2}} \\ &\leq A^{\frac{t}{2}}A^{1-t}A^{\frac{t}{2}} = A \text{ since } A^{1-t} \geq B^{1-t} \text{ by } (*) \end{aligned}$$

because the first inequality follows by (3) and $(*)$ since $1-\lambda \in [0,1]$, finally taking inverses of both sides since $\lambda - 1 \in [-1,0]$. Whence $A \geq B_1 > 0$ holds, so that we obtain $A^{1+r_1} \geq (A^{\frac{r_1}{2}}B_1^{p_1}A^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}}$ for $p_1 \geq 1$ and $r_1 \geq 0$ by Theorem A for $t = 0$ and $s = 1$. We have only to put $r_1 = r - t \geq 0$ and $p_1 = (p-t)s + t \geq 1$ to obtain (2). □

We remark that although there are many proofs of Theorem A, we cite one-page proof in [2, p. 133] and a proof of Theorem B in this paper is given along this one-page proof.

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References

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